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# *A Comitant Curve of the Plane Quartic.**

By Teresa Cohen.

## I. The Conic on a Line and the Four Tangents at its Intersections with a Plane Quartic.

If a quartic curve be cut by a line $\xi$ and the tangents to the quartic at the intersections be drawn, the four tangents together with $\xi$ itself determine a conic. This conic is to be expressed in terms of known comitants of the quartic.

For this it is necessary to know the degree of the conic both in $\xi$ and in the coefficients of the quartic. While not known directly, it may be obtained indirectly by finding the contact of $\xi$ with the conic. A line conic may be considered as the locus of lines cutting four fixed lines in a given double ratio. Now suppose two lines $\xi$ and $\xi'$ to cut the quartic in $A, B, C, D$ and $A', B', C', D'$, respectively, so that

$$(AB/CD) = (A'B'/C'D'). \tag{1}$$

Then the lines $AA', BB', CC', DD', \xi, \xi'$ touch a conic. Now allow $\xi'$ to approach $\xi$, $(A'B'/C'D')$ being kept constantly equal to $(AB/CD)$. $AA', BB', CC', DD'$ approach the tangents to the quartic at $A$, $B$, $C$, $D$, respectively, the conic approaches the required conic, and the intersection of $\xi$ and $\xi'$ approaches the contact of $\xi$ with the latter. But $\xi$ and $\xi'$, cutting the quartic in a constant double ratio, are lines of a known locus. Therefore, the contact of $\xi$ with the required conic is the same as its contact with a known curve.

A quartic $(\alpha x)^4$ has two contravariants, $(s\xi)^4$ and $(t\xi)^6$, the loci of lines cut by the quartic in a self-apolar set and in harmonic pairs, respectively.† The locus of lines cutting the quartic in a given double ratio is

$$(s\xi)^4(s'\xi)^4(s''\xi)^4 - \lambda(t\xi)^6(t'\xi)^6 = 0,$$

or, more briefly,

$$s^3 - \lambda t^2 = 0,$$

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* Presented to the American Mathematical Society, December 28, 1916.

† These are understood to be taken as given in Salmon, "Higher Plane Curves," so that in symbolic notation $(s\xi)^4 = \frac{1}{3}|\alpha\beta\xi|^2$ and $(t\xi)^6 = \frac{1}{6}|\beta\gamma\xi|^2|\gamma\alpha\xi|^2|\alpha\beta\xi|^2$.

where $\lambda$ is a constant depending on the given double ratio. The polar point of $\xi$ with respect to this curve is

$$(s\xi)^3(s\eta)\,(s'\xi)^4(s''\xi)^4-\lambda(t\xi)^5(t\eta)\,(t'\xi)^6=0,$$

or
$$s^2s_1-\lambda tt_1=0.$$

If $\xi$ is on the curve, we may eliminate $\lambda$ between these two equations. Throwing out the factor $s^2t$, since $\xi$ is in general on neither $s$ nor $t$, we obtain for the polar point, or point of contact, of $\xi$

$$(s\xi)^3(s\eta)\,(t\xi)^6-(s\xi)^4(t\xi)^5(t\eta)=0, \text{ or } s_1t-st_1=0.$$

This is of degree 5 in the coefficients of the quartic and 9 in $\xi$; therefore, the conic itself should be of degree 5 in the coefficients of the quartic and 8 in $\xi$, which may be stated symbolically by saying that the conic is an $A^5\xi^8\eta^2$.

Under certain circumstances the degree in $\xi$ will be lowered. The fact that the polar of $\xi$ is $s_1t-st_1$ indicates that if $(s\xi)^4$ or $(t\xi)^6$ has a repeated factor, this factor appears in the $A^5\xi^8\eta^2$. Also suppose the curve to have a double point and $\xi$ to be a line through it. Then $\xi$ intersects the quartic in points whose double ratio is 0, 1, $\infty$ according to the order in which they are taken. But since any line through the double point can be regarded as an improper sort of tangent, the question arises, What lines should be taken as the two tangents here? Since with this choice of $\xi$ in equation (1) $A=B$, then $A'=B'$. Then $AA'$ is the same line as $BB'$ and since the tangents at $A$ and $B$ are the limiting positions of these as $\xi'$ approaches $\xi$, then the two tangents at the double point are to be taken as coincident. But then any line $\eta$ cuts the four tangents where $\xi$ meets the quartic in the same double ratio as $\xi$ does and the conic becomes arbitrary. Therefore, the double points factor out of the $A^5\xi^8\eta^2$ at least once. The degree in $\xi$ is not high enough for them to occur more than once.

There are only five independent forms of proper order to make up the $A^5\xi^8\eta^2$. Therefore,

$$\begin{aligned}A^5\xi^8\eta^2=&\lambda(s\xi)^2(s\eta)^2(t\xi)^6+\mu(s\xi)^3(s\eta)\,(t\xi)^5(t\eta)+\nu(s\xi)^4(t\xi)^4(t\eta)^2\\&+\rho(s\xi)^3(s\alpha)\,(s'\xi)^3(s'\alpha)\,|\alpha\xi\eta|^2+\sigma(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\xi\eta|^2,\end{aligned}\qquad(2)$$

where $\lambda$, $\mu$, $\nu$, $\rho$, $\sigma$ are constant coefficients, the value of which is to be determined. Since the conic is on $\xi$,

$$\begin{aligned}\lambda(s\xi)^4(t\xi)^6+\mu(s\xi)^4(t\xi)^6+\nu(s\xi)^4(t\xi)^6=0,\\ \lambda+\mu+\nu=0.\end{aligned}\qquad(\text{I})$$

This is one relation on the coefficients. For further relations special quartics for which the conic is known must be used.

*Special Cases.*

A. The simplest quartic for which the conic is known is $4x_0(x_1^2+x_2^2)=0$, representing four lines, three of which pass through the point $(1, 0, 0)$. The conic for any line $\xi$ is made up of this point and the point where $\xi$ cuts the fourth line, $x_0=0$. Its equation is

$$\eta_0(\eta_1\xi_2-\eta_2\xi_1)=0.$$

The double points of the quartic and the repeated factors of $(s\xi)^4$ and $(t\xi)^6$ account for an outstanding factor of degree 7 in $\xi$. Therefore,

$$A^5\xi^8\eta^2=\xi_0^4(\xi_1^3-\xi_2^3)(\eta_0\eta_1\xi_2-\eta_0\eta_2\xi_1).$$

By substitution in equation (2),

$$\begin{aligned}A^5\xi^5\eta^2=&\eta_0^2(\xi_0^3\xi_1^4\xi_2+\xi_0^3\xi_1\xi_2^4)\left(8\lambda+12\mu+\tfrac{48}{5}\nu+36\rho+48\sigma\right)\\&+\overset{2}{\Sigma}\eta_0\eta_1[\xi_0^4\xi_1^3\xi_2(16\lambda+18\mu+\tfrac{144}{5}\nu-54\rho-96\sigma)+\xi_0^4\xi_2^4(16\lambda+6\mu-18\rho)]\\&+\overset{2}{\Sigma}\eta_1^2\cdot\xi_0^5\xi_1^2\xi_2(6\mu+\tfrac{48}{5}\nu+18\rho+48\sigma)+\eta_1\eta_2(\xi_0^5\xi_1^3+\xi_0^5\xi_2^3)(8\lambda+6\mu+18\rho).\end{aligned}$$

From this come the relations

$$4\lambda+3\mu+9\rho=0, \tag{II}$$

and

$$\mu+\frac{8}{5}\nu+3\rho+8\sigma=0. \tag{III}$$

B. The same relations would have been obtained by use of the four general lines

$$3[x_0^4+x_1^4+x_2^4-2x_1^2x_2^2-2x_2^2x_0^2-2x_0^2x_1^2]=0,$$

for which the conic is evidently

$$\begin{vmatrix}\eta_0^2 & \eta_1^2 & \eta_2^2\\ \xi_0^2 & \xi_1^2 & \xi_2^2\\ 1 & 1 & 1\end{vmatrix}=0.$$

By including the six double points

$$\begin{aligned}A^5\xi^8\eta^2&=(\xi_1^2-\xi_2^2)(\xi_2^2-\xi_0^2)(\xi_0^2-\xi_1^2)\overset{3}{\Sigma}\eta_0^2(\xi_1^2-\xi_2^2)\\&=\overset{3}{\Sigma}\eta_0^2(-\xi_0^4\xi_1^4+2\xi_0^4\xi_1^3\xi_2^2-\xi_0^4\xi_2^4+\xi_0^2\xi_1^6-\xi_0^2\xi_1^4\xi_2^3-\xi_0^2\xi_1^2\xi_2^4+\xi_0^2\xi_2^6-\xi_1^6\xi_2^2+2\xi_1^4\xi_2^4-\xi_1^2\xi_2^6).\end{aligned}$$

C. The lemniscate

$$6(x_1^2x_2^2+x_2^2x_0^2+x_0^2x_1^2)=0$$

is another case for which the conic can be found. Since it has three double points, $\xi_0\xi_1\xi_2$, and $(s\xi)^4=3(\xi^2)^2$, where $(\xi^2)=\xi_0^2+\xi_1^2+\xi_2^2$, the $A^5\xi^8\eta^2$ has the factor $\xi_0\xi_1\xi_2(\xi^2)$ and the conic reduces to the third degree in $\xi$.

That the conic can be found in this case is due to the property of the lemniscate that a line satisfying the equation $(\xi^2)=0$ cuts the lemniscate so

that the four tangents at the intersections meet in the point $(\eta/\xi)=0$, or $(\eta_0\xi_1\xi_2)=0$. The conic, then, breaks up into two points, of which this is one; the other, found from $s_1t-st_1$, is $(\xi\eta)=0$. Therefore, for such lines as obey the relation $(\xi^2)=0$ the conic is

$$(\xi\eta)(\eta_0\xi_1\xi_2)=0.$$

But the conic in general can differ from this only by terms containing $(\xi^2)$, and must be of the form

$$(\xi\eta)(\eta_0\xi_1\xi_2)+\varkappa(\xi^2)(\eta_1\eta_2\xi_0)=0.$$

Since it must be on $\xi$, $\varkappa=-1$. Therefore, the conic is

$$(\xi\eta)(\eta_0\xi_1\xi_2)-(\xi^2)(\eta_1\eta_2\xi_0)=0, \text{ or } \xi_0\xi_1\xi_2(\eta^2)-(\eta_1\eta_2\xi_0^2)=0.$$

It can furthermore be shown by direct proof that this conic touches the four tangents at the intersections of $\xi$ with the quartic. Let $y$ be one of these points. Then

$$(\xi y)=0, \quad \left(\frac{1}{y^2}\right)=0.$$

The tangent at $y$ is $(x/y^3)=0$; that is, its coordinates are $1/y_0^3$, $1/y_1^3$, $1/y_2^3$. Substituting these values in the equation of the conic and remembering that if $a+b+c=0$, then $a^3+b^3+c^3=3abc$, we find

$$\xi_0\xi_1\xi_2\left(\frac{1}{y^6}\right)-\left(\frac{\xi_0^3}{y_1^3y_2^3}\right)=\frac{3\xi_0\xi_1\xi_2}{y_0^2y_1^2y_2^2}-\frac{(\xi^3y^3)}{y_0^3y_1^3y_2^3}=\frac{3\xi_0\xi_1\xi_2y_0y_1y_2-3\xi_0\xi_1\xi_2y_0y_1y_2}{y_0^3y_1^3y_2^3}=0.$$

Therefore, the tangent to the quartic at $y$ is a line of the conic.

To obtain the $A^5\xi^5\eta^2$ the conic must be multiplied by the outstanding factor $(\xi^2)\xi_0\xi_1\xi_2$.

$$A^5\xi^8\eta^2=\overset{3}{\Sigma}\eta_0(\xi_0^4\xi_1^2\xi_2^2+\xi_0^2\xi_1^4\xi_2^2+\xi_0^2\xi_1^2\xi_2^4)+\overset{3}{\Sigma}\eta_1\eta_2\xi_1\xi_2(-\xi_0^6-\xi_0^4\xi_1^2-\xi_0^4\xi_2^2).$$

The relations on $\lambda$, $\mu$, $\nu$, $\rho$, $\sigma$, given by this special quartic, however, are all more simply found from the following case.

D. The type of the conic for the quartic $x_0^4+x_1^4+x_2^4=0$ can be obtained from consideration of

$$s_1t-st_1=\xi_0\xi_1\xi_2[(\xi^3\eta)\xi_0\xi_1\xi_2-\tfrac{1}{3}(\xi^4)(\eta_0\xi_1\xi_2)].$$

To give such a polar as to $\xi$ (the repeated factor $\xi_0\xi_1\xi_2$ of $(t\xi)^6$ being dropped) the conic must be of the form

$$l(\xi^2\eta^2)\xi_0\xi_1\xi_2+m(\xi^3\eta)(\eta_0\xi_1\xi_2)+n(\xi^4)(\eta_1\eta_2\xi_0),$$

where $l$, $m$, $n$ are to be determined. Since the conic is on $\xi$,

$$l+3m+3n=0.$$

For further relations on $l, m, n$ take a special line, say through the reference point $\xi_0=0$. The line is then of the form $\varkappa x_1+x_2=0$, and it meets the curve in four points given by

$$x_0^4+(1+\varkappa^4)x_1^4=0.$$

Let $\alpha$ be any one of the fourth roots of $-(1+\varkappa)^4$. Then the four points are

$$(\alpha, 1, -\varkappa),\ (-\alpha, 1, -\varkappa),\ (i\alpha, 1, -\varkappa),\ (-i\alpha, 1, -\varkappa);$$

the four tangents to the curve at these points are

$$\alpha^3x_0+x_1-\varkappa^3x_2=0,\ -\alpha^3x_0+x_1-\varkappa^3x_2=0,\ -i\alpha^3x_0+x_1-\varkappa^3x^4=0,\ i\alpha^3x_0+x_1-\varkappa^3x_2=0.$$

These are all satisfied by the point $(0, \varkappa^3, 1)$; for the line $\varkappa x_1+x_2=0$, $s_1t-st_1=\eta_0$. Then for the line $(0, \varkappa, 1)$

$$m(\varkappa^3\eta_1+\eta_2)\cdot\varkappa\eta_0+n(\varkappa^4+1)(\eta_0\eta_1+\varkappa\eta_0\eta_2)\equiv\eta_0(\varkappa^3\eta_1+\eta_2).$$

$$\therefore\ n=0,\ m=-1,\ l=3.$$

The conic is

$$3(\xi^2\eta^2)\xi_0\xi_1\xi_2-(\xi^3\eta)(\eta_0\xi_1\xi_2)=0.$$

By a method similar to that for the lemniscate it can be shown satisfied by $(xy^3)=0$, the tangent at $y$, when $(y^4)=0$, $(\xi y)=0$. To obtain the $A^5\xi^8\eta^2$ multiply by $\xi_0\xi_1\xi_2$, the repeated factor of $(t\xi)^6$.

$$A^5\xi^8\eta^2=\sum^3\eta_0^2\cdot 2\xi_0^4\xi_1^2\xi_2^2+\sum^3\eta_1\eta_2\xi_1\xi_2(-\xi_0^2\xi_1^4-\xi_0^2\xi_2^4).$$

Substituting in (2),

$$A^5\xi^8\eta^2=\sum^3\eta_0^2[\xi_0^4\xi_1^2\xi_2^2(\lambda+\tfrac{1}{3}\mu+\tfrac{1}{15}\nu+2\sigma)+(\xi_1^6\xi_2^2+\xi_1^2\xi_2^6)(\tfrac{1}{15}\nu+\rho+2\sigma)]$$
$$+\sum^3\eta_1\eta_2\xi_1\xi_2[(\xi_0^2\xi_1^4+\xi_0^2\xi_2^4)(\tfrac{1}{3}\mu+\tfrac{4}{15}\nu-2\sigma)+\xi_0^6(\tfrac{4}{15}\nu-2\rho-2\sigma)].$$

Then
$$\tfrac{1}{15}\nu+\rho+2\sigma=0. \qquad \text{(IV)}$$

This equation together with (I), (II) and (III) is sufficient to determine $\lambda, \mu, \nu, \rho, \sigma$ (or rather their ratios).

$$\lambda=0,\ \mu=15,\ \nu=-15,\ \rho=-5,\ \sigma=3.$$

E. The formula

$$A^5\xi^8\eta^2=15(s\xi)^3(s\eta)(t\xi)^5(t\eta)-15(s\xi)^4(t\xi)^4(t\eta)^2-5(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)|\alpha\xi\eta|^2$$
$$+3(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\xi\eta|^2$$

may be tested on the general quartic

$$ax_0^4+4a_1x_0^3x_1+4a_2x_0^3x_2+6hx_0^2x_1^2+12lx_0^2x_1x_2+6gx_0^2x_2^2+4b_0x_0x_1^3+12mx_0x_1^2x_2$$
$$+12nx_0x_1x_2^2+4c_0x_0x_2^3+bx_1^4+4b_2x_1^3x_2+6fx_1^2x_2^2+4c_1x_1x_2^3+cx_2^4=0 \qquad (3)$$

for a special reference triangle and line $\xi$. Let the line $\xi$ be $x_0=0$ and let two of the tangents at its intersections be $x_1=0$ and $x_2=0$. This requires that

$$b=c=b_0=c_0=0.$$

Then beside the two reference points $(0,1,0)$ and $(0,0,1)$ the line $x_0=0$ cuts the quartic in two points given by

$$4b_2x_1^2+6fx_1x_2+4c_1x_2^2=0. \tag{4}$$

Let these two points be $(0,A,1)$ and $(0,B,1)$. Then the tangent at $(0,A,1)$ is

$$x_0(3mA^2+3nA)+x_1(3b_2A^2+3fA+c_1)+x_2(b_2A^3+3fA^2+3c_1A)=0,$$

or, making use of the fact that $(0,A,1)$ must satisfy (4),

$$x_0(3mA+3n)+x_1(2b_2A+\tfrac{3}{2}f)+x_2(\tfrac{3}{2}fA+2c_1)=0. \tag{5}$$

Similarly, the tangent at $(0,B,1)$ is

$$x_0(3mB+3n)+x_1(2b_2B+\tfrac{3}{2}f)+x_2(\tfrac{3}{2}fB+2c_1)=0. \tag{6}$$

The conic, being on the reference lines, must be of the form

$$\alpha_0\eta_1\eta_2+\alpha_1\eta_2\eta_0+\alpha_2\eta_0\eta_1=0.$$

Requiring it to be satisfied by the lines (5) and (6) gives two homogeneous equations in $\alpha_0, \alpha_1, \alpha_2$, which can be solved for them in terms of $A$ and $B$. But $A$ and $B$ enter symmetrically, so that they can be gotten rid of by use of symmetric functions of the roots of (4). The conic is obtained in the form

$$\eta_0\eta_1(-4b_2c_1n+3c_1fm)+\eta_0\eta_2(-4b_2c_1m+3b_2fn)+\eta_1\eta_2(6b_2n^2+6c_1m^2-9fmn)=0,$$

when the factor $-16b_2c_1+9f^2$ has been dropped.

By use of equation (2)*

$$\begin{aligned}A^5\xi^3\eta^2=\eta_0^2[&b_2^2c_1^2f(-8\mu-8\nu\equiv 0)\\&+b_2c_1f^3(10\mu+10\nu\equiv 0)\\&+f^5(-3\mu-3\nu\equiv 0)]\\+\sum^2[&2\eta_0\eta_1[b_2^2c_1^2n(\tfrac{4}{3}\mu+\tfrac{8}{3}\nu\equiv -20)\\&+b_2c_1^2fm(5\mu+4\nu\equiv 15)\\&+b_2c_1f^2n(-6\mu-6\nu\equiv 0)\\&+c_1f^3m(-3\mu-3\nu\equiv 0)\\&+f^4n(3\mu+3\nu\equiv 0)]\end{aligned}$$

* For the coefficients of $s$ and $t$ see the note at the end of the article.

$$
\begin{aligned}
&+\overset{2}{\Sigma}\eta_1^2[b_2^2c_1^2g(-\tfrac{8}{15}\nu+16\rho+24\sigma\equiv 0)\\
&\quad+b_2c_1^3h(\tfrac{24}{15}\nu+8\sigma\equiv 0)\\
&\quad+b_2c_1^2fl(-\tfrac{24}{15}\nu-8\sigma\equiv 0)\\
&\quad+b_2c_1^2mn(-2\mu-\tfrac{48}{15}\nu-6\rho-16\sigma\equiv 0)\\
&\quad+b_2c_1f^2g(\tfrac{18}{15}\nu-24\rho-34\sigma\equiv 0)\\
&\quad+c_1^2f^2h(-\tfrac{18}{15}\nu-6\sigma\equiv 0)\\
&\quad+b_2c_1fn^2(2\mu+\tfrac{48}{15}\nu+6\rho+16\sigma\equiv 0)\\
&\quad+c_1^2fm^2(-3\mu-9\rho\equiv 0)\\
&\quad+c_1f^3l(\tfrac{18}{15}\nu+6\sigma\equiv 0)\\
&\quad+c_1f^2mn(6\mu+\tfrac{36}{15}\nu+18\rho+12\sigma\equiv 0)\\
&\quad+f^4g(-\tfrac{9}{15}\nu+9\rho+12\sigma\equiv 0)\\
&\quad+f^3n^2(-3\mu-\tfrac{36}{15}\nu-9\rho-12\sigma\equiv 0)]\\
&+2\eta_1\eta_2[b_2^2c_1^2l(-\tfrac{16}{15}\nu-16\rho-32\sigma\equiv 0)\\
&\qquad+(b_2^2c_1fg+b_2c_1^2fh)(-\tfrac{12}{15}\nu-4\sigma\equiv 0)\\
&\qquad+(b_2^2c_1n^2+b_2c_1^2m^2)(-\mu-\tfrac{24}{15}\nu+15\rho+32\sigma\equiv 30)\\
&\qquad+b_2c_1f^2l(\tfrac{36}{15}\nu+24\rho+52\sigma\equiv 0)\\
&\qquad+b_2c_1fmn(-\mu+\tfrac{12}{15}\nu-30\rho-56\sigma=-45)\\
&\qquad+(b_2f^3g+c_1f^3h)(\tfrac{9}{15}\nu+3\sigma\equiv 0)\\
&\qquad+(b_2f^2n^2+c_1f^2m^2)(3\mu+\tfrac{18}{15}\nu-9\rho-24\sigma\equiv 0)\\
&\qquad+f^4l(-\tfrac{18}{15}\nu-9\rho-21\sigma\equiv 0)\\
&\qquad+f^3mn(-3\mu-\tfrac{9}{15}\nu+18\rho+42\sigma\equiv 0)]\\
&=5b_2c_1[\overset{2}{\Sigma}2\eta_0\eta_1(-4b_2c_1n+3c_1fm)+2\eta_1\eta_2(6b_2n^2+6c_1m^2-9fmn)].
\end{aligned}
$$

Therefore, *the conic on a line $\xi$ and the tangents at its four intersections with a quartic is given by*

$$
\begin{aligned}
&15(s\xi)^3(s\eta)(t\xi)^5(t\eta)-15(s\xi)^4(t\xi)^4(t\eta)^2-5(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)|\alpha\xi\eta|^2\\
&\qquad+3(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\xi\eta|^2.
\end{aligned}
$$

## II. The Locus of Lines Meeting the Quartic so that Three of the Tangents at the Intersections Are on a Point.

For certain lines $\xi$ the conic breaks up. These are the lines such that three of the tangents at the intersections with the quartic are on a point. The discriminant of the $A^5\xi^8\eta^2$, an $A^{15}\xi^{24}$, is therefore the locus of such lines. The line equation of the quartic does not enter into this $A^{15}\xi^{24}$, as may be seen from case E above, where $16b_2c_1-9f^2$, the condition that $x_0=0$ be tangent to the quartic, is not a factor in the discriminant of the conic.

The expression obtained for the $A^5\xi^8\eta^2$ is of the form

$$\begin{aligned}&(a_1x_0^2+b_1x_1^2+c_1x_2^2+2f_1x_1x_2+2g_1x_2x_0+2h_1x_0x_1)\\+&(a_2x_0^2+b_2x_1^2+c_2x_2^2+2f_2x_1x_2+2g_2x_2x_0+2h_2x_0x_1)\\+&(a_3x_0^2+b_3x_1^2+c_3x_2^2+2f_3x_1x_2+2g_3x_2x_0+2h_3x_0x_1)\\+&(a_4x_0^2+b_4x_1^2+c_4x_2^2+2f_4x_1x_2+2g_4x_2x_0+2h_4x_0x_1).\end{aligned}$$

Therefore, if

$$\Delta_{iii}=\begin{vmatrix}a_i&h_i&g_i\\h_i&b_i&f_i\\g_i&f_i&c_i\end{vmatrix},\quad \Delta_{iij}=\sum^{3}\begin{vmatrix}a_i&h_i&g_j\\h_i&b_i&f_j\\g_i&f_i&c_j\end{vmatrix},\quad \Delta_{ijk}=\sum^{6}\begin{vmatrix}a_i&h_j&g_k\\h_i&b_j&f_k\\g_i&f_j&c_k\end{vmatrix},$$

then

$$A^{15}\xi^{24}=\begin{vmatrix}a_1+a_2+a_3+a_4&h_1+h_2+h_3+h_4&g_1+g_2+g_3+g_4\\h_1+h_2+h_3+h_4&b_1+b_2+b_3+b_4&f_1+f_2+f_3+f_4\\g_1+g_2+g_3+g_4&f_1+f_2+f_3+f_4&c_1+c_2+c_3+c_4\end{vmatrix}$$
$$=\sum^{4}\Delta_{111}+\sum^{12}\Delta_{112}+\sum^{4}\Delta_{123},$$

where

$$\Delta_{111}=0,\quad \Delta_{333}=0,\quad \Delta_{444}=0,$$

$$\Delta_{222}=-\tfrac{15^3}{6}(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2,$$

$$\Delta_{112}=-\tfrac{15^3}{24}\{(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2$$
$$-3(s\xi)^4(s'\xi)^3(s''\xi)^3(t\xi)^6(t'\xi)^4(t''\xi)^4|s't't''||s''t't''|\},$$

$$\Delta_{122}=\tfrac{15^3}{6}(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2,$$

$$\Delta_{113}=\tfrac{15^3\cdot5}{4}(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)\{(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^6$$
$$-2(s''\xi)^4(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^5(t'\alpha)$$
$$+(s''\xi)^4(s'''\xi)^4(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},$$

$$\Delta_{133}=\tfrac{15\cdot5^2}{2}(s\xi)^4(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha')(s^{iv}\xi)^3(s^{iv}\alpha')|\alpha\alpha'\xi|^2(t\xi)^6,$$

$$\Delta_{114}=-\tfrac{15^2\cdot3}{4}(s\xi)^4(s'\xi)^2(s'\alpha)^2\{(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^6$$
$$-2(s''\xi)^4(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^5(t'\alpha)$$
$$+(s''\xi)^4(s'''\xi)^4(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},$$

$$\Delta_{144}=\tfrac{15\cdot3^2}{2}(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2(s^{iv}\xi)^2(s^{iv}\alpha')^2|\alpha\alpha'\xi|^2(t\xi)^6,$$

$$\Delta_{223}=-15^2.5(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)$$
$$\{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},$$

$$\Delta_{233}=-\tfrac{15\cdot5^2}{2}(s\xi)^4(s'\xi)^3(s'\alpha)(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha')(s^{iv}\xi)^3(s^{iv}\alpha')|\alpha\alpha'\xi|^2(t\xi)^6,$$

$$\Delta_{224}=15^2.3(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2$$
$$\{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},$$

$$\Delta_{244}=-\tfrac{15\cdot3^2}{2}(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2(s^{iv}\xi)^2(s^{iv}\alpha')^2|\alpha\alpha'\xi|^2(t\xi)^6,$$

$$\Delta_{334}=0,\quad \Delta_{344}=0,$$

$$\Delta_{123}=15^2.5(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)$$
$$\{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},$$

$$\Delta_{124}=-15^2.3(s\xi)^4(s'\xi)^4(s''\xi)^4(s'''\xi)^2(s'''\alpha)^2$$
$$\{(t\xi)^6(t'\xi)^4(t'\alpha)^2-(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\},$$

$$\Delta_{134} = -15^2(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(s^{iv}\xi)^2(s^{iv}\alpha)^2|\alpha\alpha'\xi|^2(t\xi)^6,$$
$$\Delta_{234} = 15^2(s\xi)^4(s'\xi)^4(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(s^{iv}\xi)^2(s^{iv}\alpha)^2|\alpha\alpha'\xi|^2(t\xi)^6.$$

Then for convenience dropping the factor $-\frac{15^2}{8}$,

$$\begin{aligned} A^{15}\xi^{24} = {} & 5(s\xi)^4(s'\xi)^4(s''\xi)^4(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2 \\ & -15(s\xi)^4(s'\xi)^3(s''\xi)^3(t\xi)^6(t'\xi)^4(t''\xi)^4|s't't''||s''t't''| \\ & -2\{5(s\xi)^3(s\alpha)(s'\xi)^3(s'\alpha)-3(s\xi)^4(s'\xi)^2(s'\alpha)^2\} \\ & \quad \{(s''\xi)^3(s''\alpha)(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^6 \\ & \quad -2(s''\xi)^4(s'''\xi)^3(s'''\alpha)(t\xi)^6(t'\xi)^5(t'\alpha) \\ & \quad +(s''\xi)^4(s'''\xi)^4(t\xi)^5(t\alpha)(t'\xi)^5(t'\alpha)\}. \end{aligned}$$

*Special Cases.*

The correctness of this formula may be tested on the special cases given before for which the conic is known.

A. For the quartic $4x_0(x_1^3+x_2^3)=0$

$$A^{15}\xi^{24} \equiv 0.$$

B. For the four lines $3[x_0^4+x_1^4+x_2^4-2x_1^2x_2^2-2x_2^2x_0^2-2x_0^2x_1^2]=0$

$$A^{15}\xi^{24} = (\xi_1^2-\xi_2^2)^4(\xi_2^2-\xi_0^2)^4(\xi_0^2-\xi_1^2)^4.$$

C. For the lemniscate $6[x_1^2x_2^2+x_2^2x_0^2+x_0^2x_1^2]=0$

$$A^{15}\xi^{24} = 180\cdot 162^2(\xi^2)^4\xi_0^4\xi_1^4\xi_2^4[(\xi^2)^2-3(\xi_1^2\xi_2^2)].$$

D. For the quartic $x_0^4+x_1^4+x_2^4=0$

$$A^{15}\xi^{24} = \tfrac{10}{3}\xi_0^4\xi_1^4\xi_2^4[(\xi^4)(\xi_1^4\xi_2^4)-9\xi_0^4\xi_1^4\xi_2^4].$$

In each case the result obtained by the formula agrees with that obtained directly from the $A^5\xi^8\eta^2$.

It should be noted that the double points and repeated factors of $s$ and $t$, which occur *once* in the $A^5\xi^8\eta^2$ and therefore would be expected to enter the $A^{15}\xi^{24}$ *three* times, actually occur *four* times.

*Singular Lines of the Locus.*

Of special interest in connection with the $A^{15}\xi^{24}$ are the lines $\xi$ such that

$$(s\xi)^3(s\eta)(t\xi)^6-(s\xi)^4(t\xi)^5(t\eta)\equiv 0;$$

that is, the contact of $\xi$ with the $A^5\xi^8\eta^2$ is indeterminate. This occurs 1) if $s$ or $t$ has a repeated factor; 2) for the common lines of $s$ and $t$, the stationary lines of the quartic; 3) for the lines having the same polar point as to $s$ and $t$.

1) For the general quartic neither $s$ nor $t$ has repeated factors. Therefore, this case need not be discussed here.

2) It is evident that the common lines of $s$ and $t$ are double lines of the

$A^{15}\xi^{24}$, for every term of the latter contains either $s^2$, $st$, or $t^2$. Furthermore, the polar conic when $s=t=0$ becomes

$$(s\xi)^3(s\eta)\,(s'\xi)^3(s'\eta)\cdot -\tfrac{20}{3}(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha)\,(t\xi)^5(t\alpha)\,(t'\xi)^5(t'\alpha)$$
$$+(s\xi)^3(s\eta)\,(t\xi)^5(t\eta)\,[20\,(s'\xi)^3(s'\alpha)\,(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha)\,(t'\xi)^5(t'\alpha)$$
$$-15\,(s'\xi)^3(s''\xi)^3(t'\xi)^4(t''\xi)^4|s't't''|\,|s''t't''|\,]$$
$$+(t\xi)^5(t\eta)\,(t'\xi)^5(t'\eta)\cdot -15\,(s\xi)^3(s\alpha)\,(s'\xi)^3(s'\alpha)\,(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha).$$

But $(s''\xi)^3(s''\alpha)\,(s'''\xi)^3(s'''\alpha)\,(t\xi)^5(t\alpha)\,(t'\xi)^5(t'\alpha)=0$ is the condition that the point $(s\xi)^3(s\eta)$ lie on the polar conic as to the quartic of $(t\xi)^5(t\eta)$, a condition known to be fulfilled when $\xi$ is a stationary line of the quartic, since $(t\xi)^5(t\eta)$ is the flex itself. Therefore, the coefficient of $(s\xi)^3(s\eta)\,(s'\xi)^3(s'\eta)$ vanishes, showing that $(t\xi)^5(t\eta)$ is one of the points of contact of $\xi$ with the $A^{15}\xi^{24}$. Therefore $\xi$ is not only a double line of both the $A^{15}\xi^{24}$ and the quartic $(ax)^4$, but also has one point of contact the same in each case, so that the stationary lines account for $24(2\cdot 2+1)=120$ of the common lines of the two curves.

3) For a line $\xi$ to have the same polar point as to $s$ and $t$ requires that

$$\frac{(s\xi)^3 s_0}{(t\xi)^5 t_0}=\frac{(s\xi)^3 s_1}{(t\xi)^5 t_1}=\frac{(s\xi)^3 s_2}{(t\xi)^5 t_2}.$$

These equations have $25+9+15=49$ solutions. Among these must be included the twenty-eight double lines of the quartic, for both $s_1t-st_1\equiv 0$ and $s^2s_1-27tt_1\equiv 0$ hold if $s^3-27t^2=0$, which is true for the double lines. The remaining twenty-one lines are those which meet the quartic so that all four tangents at the intersections are on a point; they correspond to the twenty-one double points of the Steinerian, which are points whose polar cubic is a conic and a line. For all of these lines the $A^5\xi^8\eta^2$ is degenerate, so that they are at least lines of the $A^{15}\xi^{24}$. But presumably they are more than this, and indeed the twenty-one lines are at least triple lines, for they cause the $A^5\xi^8\eta^2$ to vanish identically. If these lines are double lines, their polar points as to the $A^{15}\xi^{24}$ should vanish. But the polar point of $\xi$ when $s_1t-st_1\equiv 0$ becomes

$$-\tfrac{5}{12}(s'\xi)^4(s''\xi)^4[\,(t\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2(s\xi)^3(s\eta)$$
$$-(s\xi)^4(t'\xi)^4(t''\xi)^4|tt't''|^2(t\xi)^3(t\eta)\,],$$

and it is not evident that this vanishes. Furthermore, though it can be shown that $(t'\xi)^4(t''\xi)^4|t't''x|^2$ is a numerical multiple of

$$9\,(s\xi)^2(s\alpha)^2(\alpha x)^2(t\xi)^6-\tfrac{15}{4}(s\xi)^4(t\xi)^4(t\alpha)^2(\alpha x)^2$$
$$+10\,(s\xi)^3(s\alpha)\,(s'\xi)^3(s'\alpha)\,|\alpha\alpha'\xi|^2(\alpha' x)^2-\tfrac{39}{8}(s\xi)^4(s'\xi)^2(s'\alpha)^2|\alpha\alpha'\xi|^2(\alpha x)^2$$
$$+(x\xi)\,[-18\,(s\xi)^2(s\alpha)^2(t\xi)^5(t\alpha)\,(\alpha x)+\tfrac{75}{7}(s\xi)^4(t\xi)^3(t\alpha)^3(\alpha x)\,]$$
$$+(x\xi)^2[\tfrac{17}{6}(s\alpha)^4(t\xi)^6-\tfrac{21}{2}(s\xi)^2(s\alpha)^2(t\xi)^4(t\alpha)^2+\tfrac{50}{7}(s\xi)^3(s\alpha)\,(t\xi)^3(t\alpha)^3$$
$$-\tfrac{145}{56}(s\xi)^4(t\xi)^2(t\alpha)^4],$$

the substitution of this value does not give an expression that vanishes on its face.

Let us first consider the double lines for the general quartic (3) with a convenient reference triangle. Let $x_0 = 0$ be a double line with contacts $(0,1,0)$ and $(0,0,1)$, and let $(1,0,0)$ be the polar point of $x_0$ as to both $s$ and $t$.* This reference scheme requires that in the equation of the quartic

$$b=c=b_2=c_1=m=n=0.$$

Then (giving only the highest powers of $\xi_0$)

$$\begin{aligned}A^5\xi^8\eta^2 = {}&\eta_0^2[270b_0c_0f^3\xi_0^6\xi_1\xi_2+\dots.]+\eta_1^2[-90b_0c_0^2f^2\xi_0^7\xi_1+\dots.]\\&+\eta_2^2[-90b_0^2c_0f^2\xi_0^7\xi_2+\dots.]\\&+2\eta_1\eta_2[-45b_0c_0f^3\xi_0^8+135c_0f^3h\xi_0^7\xi_1+135b_0f^3g\xi_0^7\xi_2+\dots.]\\&+2\eta_0\eta_2[-45b_0c_0f^3\xi_0^7\xi_1+\dots.]+2\eta_0\eta_1[-45b_0c_0f^3\xi_0^7\xi_2+\dots.],\end{aligned}$$

so that

$$A^{15}\xi^{24} = -90b_0^3c_0^3f^9\xi_0^{22}\xi_1\xi_2 + \text{lower powers in } \xi_0.$$

Therefore, the double lines of the quartic are also double lines of the $A^{15}\xi^{24}$ with the same contacts. They then account for $28(2\cdot2+2)=168$ common lines of the two curves, and since it has already been shown that the stationary lines account for 120, all of the $24\times12=288$ common lines of the $A^{15}\xi^{24}$ and the quartic (in lines an $A^6\xi^{12}$) are accounted for.

For the twenty-one lines giving four tangents meeting in a point (which point is the common polar point of the line as to $s$ and $t$) choose a reference scheme where $x_0=0$ is one of these lines with two of its intersections as $(0,1,0)$ and $(0,0,1)$ and let the tangents at these two points be $x_2=0$ and $x_1=0$, respectively, so that $(1,0,0)$ is the meeting point of the four tangents. Then

$$b=c=b_0=c_0=m=n=0.$$

$$\begin{aligned}A^5\xi^8\eta^2 = {}&\eta_0^2[\text{terms beginning with } \xi_0^6\xi_1^2]\\&+\eta_1^2[\xi_0^6\xi_1\xi_2(120b_2^2c_1^2g-360b_2c_1^2gh-180b_2c_1fgl+270c_1f^2gh)\\&\qquad+\xi_0^6\xi_2^2(240b_2^2c_1gl-90b_2^2fg^2-90b_2c_1fgh)+\dots.]\\&+\eta_2^2[\xi_0^6\xi_1^2(240b_2c_1^2hl-90b_2c_1fgh-90c_1^2fh^2)\\&\quad+\xi_0^6\xi_1\xi_2(-360b_2^2c_1gh+120b_2c_1^2h^2-180b_2c_1fhl+270b_2f^2gh)+\dots.]\\&2\eta_1\eta_2[\xi_0^6\xi_1^2(30b_2^2c_1g^2-120b_2c_1^2gh+90c_1^3h^2+120b_2c_1^2l^2-90b_2c_1fgl\\&\qquad-180c_1^2fhl+135c_1f^2gh)\\&\quad+\xi_0^6\xi_1\xi_2(-120b_2^2c_1gl-120b_2c_1^2hl-90b_2^2fg^2+540b_2c_1fgh\\&\qquad-90c_1^2fh^2-180b_2c_1fl^2+270b_2f^2gl+27c_1f^2hl\\&\qquad-261f^3gh-168f^2l^2)\\&\quad+\xi_0^6\xi_2^2(90b_2^3g^2-120b_2^2c_1gh+30b_2c_1^2h^2+120b_2^2c_1l^2-180b_2^2fgl\\&\qquad-90b_2c_1fhl+135b_2f^2gh)+\dots.]\\&+2\eta_0\eta_2[\xi_0^7\xi_1(40b_2^2c_1^2l-15b_2^2c_1fg-15b_2c_1^2fh)\\&\qquad+\xi_0^7\xi_2(-60b_2^3c_1g+20b_2^2c_1^2h-30b_2^2c_1fl+45b_2^2f^2g)+\dots.]\\&+2\eta_0\eta_1[\xi_0^7\xi_1(20b_2^2c_1^2g-60b_2c_1^3h-30b_2c_1^2fl+45c_1^2f^2h)\\&\qquad+\xi_0^7\xi_2(40b_2^2c_1^2l-15b_2^2c_1fg-15b_2c_1^2fh)+\dots.].\end{aligned}$$

* It is evident that this polar point can not lie on the double line itself, for then the latter would have to be a line of both $s$ and $t$, all of whose common lines are accounted for by the stationary lines of the quartic.

From this it is evident that $\xi_0^{20}$ is the highest power of $\xi_0$ that can enter into the $A^{15}\xi^{24}$. Therefore, the twenty-one lines are quadruple lines of the locus. There is no reason for supposing that the $A^{15}\xi^{24}$ has any more singular lines.

NOTE.

$$(s\xi)^4 = A\xi_0^4+B\xi_1^4+C\xi_2^4+4A_1\xi_0^3\xi_1+4A_2\xi_0^3\xi_2+4B_0\xi_1^3\xi_0+4B_2\xi_1^3\xi_2+4C_0\xi_2^3\xi_0+4C_1\xi_2^3\xi_1 \\ +6F\xi_1^2\xi_2^2+6G\xi_2^2\xi_0^2+6H\xi_0^2\xi_1^2+12L\xi_0^2\xi_1\xi_2+12M\xi_0\xi_1^2\xi_2+12N\xi_0\xi_1\xi_2^2,$$

where

$$A = bc-4b_2c_1+3f^2,$$
$$A_1 = -cb_0+b_2c_0+3c_1m-3fn, \quad A_2 = -bc_0+b_0c_1+3b_2n-3fm,$$
$$F = af-2a_1n-2a_2m+gh+2l^2,$$
$$L = b_0c_0-b_2g-c_1h+2fl-mn.$$

$$(t\xi)^6 = \overset{3}{\Sigma}A_{600}\xi_0^6+6\overset{6}{\Sigma}A_{510}\xi_0^5\xi_1+15\overset{6}{\Sigma}A_{420}\xi_0^4\xi_1^2+30\overset{3}{\Sigma}A_{411}\xi_0^4\xi_1\xi_2+20\overset{3}{\Sigma}A_{330}\xi_0^3\xi_1^3 \\ +60\overset{6}{\Sigma}A_{321}\xi_0^3\xi_1^2\xi_2+90A_{222}\xi_0^2\xi_1^2\xi_2^2,$$

where

$$A_{600} = bcf-bc_1^2-cb_2^2+2b_2c_1f-f^3,$$
$$A_{510} = \tfrac{1}{3}[-bcn+bc_0c_1-2cb_0f+3cb_2m+2b_0c_1^2-b_2c_0f-2b_2c_1n-3c_1fm+3f^2n],$$
$$A_{420} = \tfrac{1}{15}[bcg-bc_0^2+8cb_0n-6cb_2l+6cfh-9cm^2-8b_0c_0c_1+2b_2c_1g-6c_1^2h \\ +4b_2c_0n+6c_2fm+6c_1fl+12c_1mn-3f^2g-12fn^2],$$
$$A_{411} = \tfrac{1}{15}[2bcl-3bc_1g-3cb_2h+bc_0n+cb_0m+9b_0c_0f-10b_0c_1n-10b_2c_0m \\ +4b_2c_1l+3b_2fg+3c_1fh+6b_2n^2+6c_1m^2-6f^2l-3fmn],$$
$$A_{330} = \tfrac{1}{10}[ca_2b_2-2ca_1f-2cb_0g-6chn+9clm+2a_1c_1^2+2b_0c_0^2-a_2c_1f-b_2c_0g \\ +6c_0c_1h-3c_0fl-3c_1gm-6c_0mn-6c_1ln+6fgn+4n^3],$$
$$A_{321} = \tfrac{1}{30}[-bca_2+3ca_1b_2+bc_0g-5cb_0l+3chm-2a_2b_2c_1-3a_1c_1f+11b_0c_1g \\ +8b_2c_0l-6b_0c_0n+3a_2f^2-15c_0fh+12c_1hn-9b_2gn+12c_0m^2-18c_1lm \\ -3fgm+15fln-6mn^2],$$
$$A_{222} = \tfrac{1}{90}[abc+2ab_2c_1+2ba_2c_0+2ca_1b_0-3af^2-3bg^2-3ch^2-20a_1b_2c_0 \\ -20a_2b_0c_1+24a_1c_1m+24a_2b_2m+24b_0c_0l-6a_1fn-6a_2fm-6b_0gn \\ -6b_2gl-6c_0hm-6c_1hl+48fgh-30fl^2-30gm^2-30hn^2+48lmn].$$

All the other coefficients can be found by symmetry from those given here.

JOHNS HOPKINS UNIVERSITY, *October*, 1916.

# *On Two Related Transformations of Space Curves.**

By William Caspar Graustein

## *The Transformation C.*

1. Two curves, corresponding point for point so that the tangents in corresponding points are parallel, are said to be related by a transformation of Combescure.† The general curve in 1—1 correspondence with a given space curve, $x=x(s)$,‡ may be given by $y=x+f\alpha+\phi\beta+\psi\gamma$, where $f, \phi, \psi$ are functions of $s$. The conditions that the tangents in corresponding points be parallel are

$$\frac{d\phi}{ds}+\frac{f}{R}+\frac{\psi}{T}=0, \quad \frac{d\psi}{ds}-\frac{\phi}{T}=0.$$

If there is set $\psi=\dfrac{dC}{ds_\gamma}$, then $\phi=-\dfrac{d^2C}{ds_\gamma^2}$, $f=\dfrac{d}{ds_\alpha}\left(C+\dfrac{d^2C}{ds_\gamma^2}\right)$, and

$$y=x+\delta, \text{ where } \delta=\frac{d}{ds_\alpha}\left(C+\frac{d^2C}{ds_\gamma^2}\right)\alpha-\frac{d^2C}{ds_\gamma^2}\beta-\frac{dC}{ds_\gamma}\gamma. \tag{1}$$

This $y$-curve is the general curve in Combescurian correspondence with the $x$-curve. Since it is determined, when the function $C$ (except for an additive constant) is given, and conversely, one may speak of the correspondence, or transformation, $C$. If $C=R$, the $y$-curve is the second polar curve of the $x$-curve (the first polar curve of a curve is the edge of regression of the polar developable, the second is the polar curve of the first, etc.). Now

$$\frac{dy}{ds}=\left(1+\frac{P(C)}{R}\right)\alpha, \text{ where } P(C)=\frac{d^2}{ds_\alpha^2}\left(C+\frac{d^2C}{ds_\gamma^2}\right)+\frac{d^2C}{ds_\gamma^2}.$$

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* Read before the American Mathematical Society, December 28, 1915.

† Aoust discusses this transformation in his "Analyse infinitesimale des courbes dans l'espace," 1876, Ch. XVI. It comes by its present title, the writer believes, from its similarity with the Combescurian transformation of triply orthogonal systems of surfaces. Salkowski gives a geometrical treatment of it in the *Mathematische Annalen*, Vol. LXVI (1909), and Sannia gives an analytical one, by means of the moving trihedral, in the *Rendiconti del Circolo Matematico di Palermo*, Vol. XX (1905), pp. 83-92.

‡ In this paper $x_1, x_2, x_3$ are the coordinates of a point $x$; $x_1=x_1(s)$, $x_2=x_2(s)$, $x_3=x_3(s)$ is the parametric representation, in terms of the arc $s$, of the curve $x=x(s)$; $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \gamma_3$, are the direction cosines of the directions $\alpha, \beta, \gamma$ of the tangent, principal normal and binormal; $R$ and $T$ are the radii of curvature and torsion; $s_\alpha$ and $s_\gamma$ are the arcs of the tangent and binormal indicatrices. Plane curves are excluded and the work is for the real domain.

Thus $\alpha_y=\pm\alpha$, and hence $\beta_y=\pm\beta$; if the choices $\alpha_y=\alpha$, $\beta_y=\beta$ are made, it follows that $\gamma_y=\gamma$. Further

$$\frac{ds_y}{ds}=\frac{R_y}{R}=\frac{T_y}{T}=1+\frac{P(C)}{R}. \tag{2}$$

Thus the rectifying lines in corresponding points are parallel also.

The surface, $z=x+r\delta$, generated by the line $xy$, is a developable. Its edge of regression, if its exists, is given by $\eta=x-R/P(C)\,\delta$.

2. The $x$- and $y$-curves are congruent* only if $P(C)=0$. Now

$$P(C)=\left(\frac{ds_\gamma}{ds_a}\right)^2\Gamma\left(\frac{dC}{ds_\gamma}\right), \tag{3}$$

where $\Gamma(\gamma)=0$ is the result of elimination of $\alpha$ and $\beta$ from the Frenet formulae. The general solution of $\Gamma(\gamma)=0$ is then $-r(c|\gamma)$,† where $(c|c)=1$ and $r>0$; hence the general solution of $P(C)=0$ is

$$C=-r\int(c|\gamma)\,ds_\gamma, \text{ or } \frac{dC}{ds_\gamma}=-r(c|\gamma). \tag{4}$$

Substitution of this value of $C$ in (1) yields $y=x+rc$.

Thus the general transformation $C$, carrying a given curve into a congruent curve, is defined by (4) and is equivalent to a translation in the direction $c$ through the distance $r$. On the other hand, if the transformation $C$ is given, (4) becomes the defining relation for the curves congruent to their transforms under it. A parametric representation of these curves, as the curves defined by (4) and in Combescurian correspondence with an arbitrarily given curve, the $\xi$-curve, may be found by a method of Salkowski.‡ Since $\gamma=\gamma_\xi$, $ds_\gamma=(ds_\gamma)_\xi$, the first of the formulae (4) may be written

$$C=-r\int(c|\gamma_\xi)\,(ds_\gamma)_\xi. \tag{5}$$

If $C$ involves $s$, $R$ and $T$, (5) and the last of the equations

$$\frac{ds}{(ds)_\xi}=\frac{R}{R_\xi}=\frac{T}{T_\xi} \tag{6}$$

may be solved for $R$ and $T$ as functions of $s$ and the parameter of the $\xi$-curve,

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* In the *Bulletin de la Société Mathématique*, Vol. VII (1878), pp. 143–154, Aoust set up a differential equation of the curves congruent to their second polar curves, which Hoppe recognized later, *Archiv der Mathematik und Physik*, Vol. LXVI (1881), pp. 386–396, as that which the direction cosines of the binormal of a space curve satisfy, and thus easily integrated. It is this idea of Hoppe that is being used here in the case of the general transformation $C$.

† N. B. $(c|\gamma)=c_1\gamma_1+c_2\gamma_2+c_3\gamma_3$, $(c|c)=c_1^2+c_2^2+c_3^2$.

‡ *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, Vol. IV (1905), pp. 64–69.

and thus either of the remaining equations of (6) may be solved for $s$ as a function of the parameter of the $\xi$-curve. Hence

$$x=\int \frac{ds}{(ds)_\xi}\,d\xi, \text{ or } x=\int \frac{R}{R_\xi}\,d\xi, \text{ or } x=\int \frac{T}{T_\xi}\,d\xi$$

yields a parametric representation of the required curves. If, in particular, $C=R\,F\left(\frac{R}{T}\right)$, this method results in the representation

$$x=r\int \frac{\int (c|\gamma_\xi)\,(ds_\gamma)_\xi}{C_\xi}\,d\xi.$$

For $C=R$ this becomes the parametric representation of the curves congruent to their second polar curves.

The $y$-curve is similar to the $x$-curve,* if $R_y$ and $T_y$ are constant multiples of $R$ and $T$; that is, by (2), if $P(C)=kR$, where $k$ is a constant, not $-1$. By (3) this condition becomes

$$\Gamma\left(\frac{dC}{ds_\gamma}\right)=kR\left(\frac{ds_\alpha}{ds_\gamma}\right)^2.$$

The general solution of this equation is

$$\frac{dC}{ds_\gamma}=r(c|\gamma)-k(x|\gamma). \tag{7}$$

For this value of $C$ (1) may be written ($k\neq 0$)

$$y-\frac{rc}{k}=(1+k)\left(x-\frac{rc}{k}\right).$$

Thus the general transformation $C$, carrying a given curve into a similar curve, is defined by (7) and is equivalent to a stretching from the point $rc/k$ with coefficient of enlargement $1+k$. Work similar to that leading to this result yields a new representation of the general $y$-curve: The general transformation $C$, carrying a given $x$-curve into a $y$-curve such that $R_y$ and $T_y$ are given multiples of $R$ and $T-R_y/R=T_y/T=1+K$, where $K$ is a given function, $\not\equiv -1$—is defined by $dC/ds_\gamma=r(c|\gamma)-(\gamma|\int K dx)$, and the most general such $y$-curve is then $y=x+\int K dx-rc$.

The curves similar to their transforms under a given transformation $C$ are defined by (7). The problem of setting up a parametric representation for these curves is, in general, incapable of explicit solution.†

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* Salkowski discusses this problem for a curve and its second polar curve, *Archiv der Mathematik und Physik*, Vol. XIV (1908), pp. 231–239.

† For the transformation $C=R$, it depends on the solution of three simultaneous linear differential equations of the second order.

3. If the $x$- and $y$-curves are congruent, the distance $xy=d$ is constant. The converse, however, is not true;* differentation of $d^2=(\delta|\delta)$ with respect to $s$ yields the product of $2P(C)$ and

$$\frac{d}{ds_\alpha}\left(C+\frac{d^2C}{ds_\gamma^2}\right). \tag{8}$$

Thus the transformation $C=c_0+c\sin(s_\gamma+k)$, $c\neq 0$, applied to any given $x$-curve, yields a non-congruent $y$-curve at a constant distance. The representation of the curves, which transform into non-congruent curves at constant distances under a given transformation $C$, may be found by Salkowski's method. Thus the curves, which are at constant distances from their second polar curves but not congruent to them, are those whose first polar curves are of constant curvature, while they themselves are not.

Whereas constant length of $xy$ is not a sufficient condition for the congruence of the $x$- and $y$-curves, constant direction of $xy$ is. This may be shown by differentiating the direction cosines $\delta/d$ of $xy$.

The direction $xy$ is relatively fixed with respect to the trihedral in $x$ only if the factors of combination of $\alpha, \beta, \gamma$ in $\delta/d$ are constant. This is true only if the $x$-curve is a helix and $C$ is such that $dC/ds_\gamma=ke^{ms_\gamma}$. If $m=1$, the two curves are congruent helices on the same cylinder.

4. The transformations $C$, such that the point tracing one of the curves lies always in the corresponding rectifying plane of the other, are given by $dC/ds_\gamma=a$, a constant; then (1) becomes $y=x+a(\alpha R/T-\gamma)$. Thus the two curves are parallel geodesics on the developable $z$; conversely, two parallel geodesics on an arbitrary developable are related by such a transformation $C$.

The curves whose rectifying planes contain the corresponding points of their second polar curves are those for which the distance from the center of curvature to the center of the osculating sphere is constant. The only such curves, which, further, are congruent to their second polar curves are helices for which $R^2=bs$; these helices lie on cylinders with involutes of circles as directrices.

The transformations $C$, such that the normal plane of one curve contains always the corresponding point of the other, are given by (8), set equal to zero. The two curves are orthogonal trajectories of the rulings of the developable $z$; conversely, two orthogonal trajectories of the rulings of an arbitrary developable are related by such a transformation $C$. The curves whose normal

* Salkowski supposes the converse true in his geometric determination of the curves congruent to their second polar curves, *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, Vol. VI (1907), pp. 54–59.

planes contain the corresponding points of their second polar curves are those whose first polar curves are of constant curvature.

The only $y$-curve in correspondence $C$ with an $x$-curve such that $y$ always lies in the osculating plane of $x$ is the $x$-curve itself.

*The Transformation A.*

5. The transformation of Combescure finds an analytic parallel in the point-to-point correspondence of two curves, in which the tangents in corresponding points have as their common perpendicular direction that of the principal normal of the given curve and make with one another an angle whose cosine equals the ratio of the corresponding elements of arc—so that the element of arc of the given curve is the projection of that of the transformed curve. The $y$-curve is the transform of a given $x$-curve by this transformation, if $(\alpha_y|\beta)=0$, $(\alpha_y|\alpha)ds_y=ds$; hence it has the general form

$$y=x+\delta, \text{ where } \delta=-\frac{dA}{ds_\alpha}\alpha-\frac{d^2A}{ds_\alpha^2}\beta+\frac{d}{ds_\gamma}\left(A+\frac{d^2A}{ds_\alpha^2}\right)\gamma. \tag{9}$$

When the function $A$ (except for an additive constant) is given, this $y$-curve is determined, and, conversely. Thus one may speak of the correspondence, or transformation, $A$.

A comparison of (9) with (1) shows that analytically the transformation $A$ becomes the transformation $C$ if $\alpha$ and $\gamma$, $ds_\alpha$ and $ds_\gamma$, $R$ and $T$, are interchanged. This relationship between the two transformations subsists to a great extent throughout. Thus, from (9),

$$\frac{dy}{ds}=\alpha+\frac{Q(A)}{T}\gamma, \text{ where } Q(A)=\frac{d^2}{ds_\gamma^2}\left(A+\frac{d^2A}{ds_\alpha^2}\right)+\frac{d^2A}{ds_\alpha^2}; \tag{10}$$

$Q(A)$ becomes $P(C)$ by replacing $A$ by $C$ and interchanging $ds_\alpha$ and $ds_\gamma$.

If $\theta$, where $\tan\theta=Q(A)/T$, is introduced, then for the $y$-curve

$$\left.\begin{aligned}
\alpha_y&=\cos\theta\,\alpha+\sin\theta\,\gamma,\\
\sqrt{\theta'^2+M^2}\,\beta_y&=\ M\beta+\theta'(\cos\theta\,\gamma-\sin\theta\,\alpha),\\
\sqrt{\theta'^2+M^2}\,\gamma_y&=-\theta'\beta+M(\cos\theta\,\gamma-\sin\theta\,\alpha),\\
\frac{1}{R_y}&=\cos\theta\sqrt{\theta'^2+M^2},\\
\frac{1}{T_y}&=\cos\theta\,\frac{\theta'M'-M\theta''-N\theta'^2-M^2N}{\theta'^2+M^2},
\end{aligned}\right\} \tag{11}$$

where

$$M=\frac{\cos\theta}{R}+\frac{\sin\theta}{T},\quad N=\frac{\sin\theta}{R}-\frac{\cos\theta}{T},$$

and the primes denote differentiation with respect to $s$.

The ruled surface, $z=x+r\delta$, generated by the line $xy$, is a developable only when $Q(A)=0$ or $A=as_\alpha+b$. The common perpendicular direction to two neighboring rulings of the surface is parallel always to the normal plane of the $x$-curve. The $x$-curve is the line of striction on the surface only if $d^2A/ds_\alpha^2+A=\text{const.}$

6. If the $x$- and $y$-curves are to be, further, in Combescurian correspondence, it is evident from (10), that $Q(A)=0$; but then the curves are obviously congruent. Now

$$Q(A)=\left(\frac{ds_\alpha}{ds_\gamma}\right)^2\Delta\left(\frac{dA}{ds_\alpha}\right), \tag{12}$$

where $\Delta(\alpha)=0$ is the differential equation of the direction cosines of the direction $\alpha$ and thus has, as its general solution, $-r(a|\alpha)$, where $(a|a)=1$, $r>0$. Hence the general solution of $Q(A)=0$ is

$$\frac{dA}{ds_\alpha}=-r(a|\alpha). \tag{13}$$

Substitution of this value in (9) gives $y=x+ra$.

The general transformation $A$, carrying a given curve into a congruent curve, is defined by (13) and is equivalent to a translation in the direction $a$ through the distance $r$. On the other hand, if the transformation $A$ is given, (13) becomes the defining relation for the curves congruent to their transforms under it. A parametric representation of these curves may be obtained by Salkowski's method.

7. The ratio of the elements of arc of the $x$- and $y$-curves is constant, or, what is equivalent, the angle $\theta$ formed by the directions of their tangents is constant, only if $Q(A)/T(=\tan\theta)$ is constant. By (12) this condition becomes

$$\Delta\left(\frac{dA}{ds_\alpha}\right)=\tan\theta\, T\left(\frac{ds_\gamma}{ds_\alpha}\right)^2.$$

The general solution of this equation is

$$\frac{dA}{ds_\alpha}=r(a|\alpha)-\tan\theta\left(\int\gamma ds|\alpha\right). \tag{14}$$

For this value of $dA/ds_\alpha$, (9) reduces to $y=x+\tan\theta\int\gamma ds-ra$.

The corresponding elements of arc of the $x$-curve and its transform by $A$ are in constant ratio, or the directions of their corresponding tangents form a constant angle $\theta$, only when $A$ is given by (14). Work similar to that leading

to this result yields a new means of representing the general $y$-curve: The general transformation $A$, carrying a given $x$-curve into a $y$-curve whose element of arc is a given multiple of that of the $x$-curve——$ds_y = \sec\theta\, ds$, where $\theta$ is a given function——is defined by $dA/ds_a = r(a|\alpha) - (\int \tan\theta\,\gamma ds\,|\,\alpha)$, and the most general such $y$-curve is then $y = x + \int \tan\theta\,\gamma ds - ra$.

If $\theta$ is constant, the formulae (11) become

$$\alpha_y = \cos\theta\,\alpha + \sin\theta\,\gamma, \quad \beta_y = \beta, \quad \gamma_y = \cos\theta\,\gamma - \sin\theta\,\alpha,$$
$$\frac{\sec\theta}{R_y} = \frac{\cos\theta}{R} + \frac{\sin\theta}{T}, \quad \frac{\sec\theta}{T_y} = \frac{\cos\theta}{T} - \frac{\sin\theta}{R}, \quad ds_y = \sec\theta\, ds.$$

Thus, if the $y$-curve results from the $x$-curve by a transformation $A$, preserving the ratio of the elements of arc, then the principal normals of the two curves are parallel, and conversely; their rectifying lines are parallel and the elements of arc of their principal normal indicatrices are in constant ratio. Further, if the given curve is a curve of constant curvature or torsion, the transformed curve is a curve of Bertrand; conversely, a curve of Bertrand may be carried over by a suitable such transformation $A$ into a curve of constant curvature or torsion. (Thus the determination of the curves of Bertrand is reduced to the determination of the curves of constant curvature or torsion.) Finally, under such a transformation $A$, a helix goes into a helix and a circular helix into a circular helix.

The surface $z$ is, for constant $\theta$, $z = x + \rho(\tan\theta \int \gamma ds - ra)$. The curve $\rho = 1$ on it is the transform of the $x$-curve by (14). Evidently an arbitrary curve $\rho = \text{const.}$ is also such a transform. Thus the curves $\rho = \text{const.}$ have their principal normals along a ruling of the surface parallel. Conversely, if a ruled non-developable surface contains a family of curves having parallel principal normals along each ruling, these curves are the transforms of one of their number by transformations of the type (14).

8. The transformations $A$, such that the point tracing the $y$-curve lies always in the corresponding rectifying plane of the $x$-curve, are given by $dA/ds_a = a$, a constant; the $y$-curve is then $y = x - a(\alpha - T/R\,\gamma)$. Evidently the surface $z$ is the rectifying developable of the $x$-curve and thus the $x$-curve is a geodesic on it. Conversely, a geodesic on a developable may be carried into any second curve on the developable, whose tangent is perpendicular to the corresponding principal normal of the geodesic, by such a transformation $A$. The $y$-curve is also a geodesic on the developable only when the $x$-curve is a helix; the two curves are then congruent.

The transformations $A$, such that the osculating plane of the $y$-curve contains always the corresponding point of the $x$-curve, are given by

$$\frac{dA}{ds_\alpha} = c\cos(s_\alpha+k). \tag{15}$$

The $y$-curve is then $y=x+\delta$, where $\delta=-c[\cos(s_\alpha+k)\alpha-\sin(s_\alpha+k)\beta]$. The $x$-curve is an asymptotic line and also the line of striction on the surface $z=x+r\delta$. The curves $r=$const. on the surface are the curves at constant distances, along the rulings, from the $x$-curve and each of them is a transform of it by a suitable transformation (15). Conversely, if the line of striction of a ruled surface is asymptotic, the curves at constant distances from it are its transforms by transformations $A$ of the form (15). If these curves are to be asymptotic also, the line of striction must be a curve of the category defined by $dT/ds_\gamma=c\cos(s_\alpha+k)$.

The only $y$-curve in correspondence $A$ with an $x$-curve so that $y$ always lies in the normal plane of $x$ is the $x$-curve itself.

9. Here again, as in the case of the transformation $C$, a constant distance $xy=d$ is not sufficient for the congruence of the $x$- and $y$-curves. In fact, the distance $d$ is constant, and the $x$- and $y$-curves are still not congruent, in case of the special transformation (15).

On the other hand, and again as in the case of the transformation $C$, constant direction of $xy$ is a sufficient condition for congruence of the $x$- and $y$-curves.

If the direction $xy$ is relatively fixed with respect to the trihedral in $x$, the $x$-curve is a helix and the transformation is given by $dA/ds_\alpha=ke^{ms_\alpha}$, and conversely.

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# *The Space Problem of the Calculus of Variations in Terms of Angle.*

By Paul R. Rider.

## *Introduction.*

In her Göttingen thesis* Gernet treated the problem of the calculus of variations for three dimensions, considering an integral of the form

$$I=\int f\left(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}\right)dx.$$

Bliss and Mason† have developed the theory for the integral

$$I=\int f(x, y, z, x_t, y_t, z_t)\,dt,$$

where the arguments of $f$ are functions of a parameter $t$.

It is the object of this paper to develop the theory for an integral in a different form from either of these, and to use the results in considering certain generalized definitions of angle and of solid angle. The first six sections treat the general theory. In § 1 the Euler equations are derived and the form of the equations of the extremals is shown. By means of the family of extremals through a given point the Jacobi necessary condition is derived in § 2. The Hilbert invariant integral and the Weierstrass $E$-function are taken up in § 3, and the necessary conditions of Weierstrass and Legendre are given in this section and in § 4. As the $F_1$-function, which first appears in § 1, is a definite expression instead of a factor of proportionality, the Legendre condition is more easily developed and is simpler than in the Weierstrass form of the problem in three-dimensional space as developed by Bliss and Mason. However, sufficient conditions are obtained for a weak extremum only—§ 5. The so-called transversality condition that must hold when one end-point is allowed to vary along a curve is given in § 6. In § 7 a transversal surface is studied and two generalized definitions of angle are made. Incidentally the differential equations of geodesics on the transversal surface are obtained. Finally, § 8 deals with a generalization of the notion of solid angle.

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*"Untersuchungen zur Variationsrechnung," Dissertation, Göttingen (1902).

†"The Properties of Curves in Space Which Minimize a Definite Integral," *Transactions of the American Mathematical Society*, Vol. IX (1908), p. 440.

## §1. *The Euler Equations.*

Suppose that there is a curve defined by the functions

$$x=\phi(t), \quad y=\psi(t), \quad z=\chi(t). \tag{C}$$

The integral to be studied is

$$I=\int f(x, y, z, \tau, \sigma)\sqrt{x_t^2+y_t^2+z_t^2}\,dt, \tag{1}$$

where $\tau$ and $\sigma$ are defined by the equations

$$\left.\begin{aligned}\frac{x_t}{\sqrt{x_t^2+y_t^2+z_t^2}}&=\cos\sigma\cos\tau,\\ \frac{y_t}{\sqrt{x_t^2+y_t^2+z_t^2}}&=\cos\sigma\sin\tau,\\ \frac{z_t}{\sqrt{x_t^2+y_t^2+z_t^2}}&=\sin\sigma.\end{aligned}\right\} \tag{2}$$

The geometric significance of $\tau$ and $\sigma$ is evident from equations (2). If at any point on the curve $C$ the positive tangent to the curve is constructed, then $\sigma$ is the angle that this tangent makes with its own projection in the $xy$-plane, and $\tau$ is the angle that this projection makes with the $x$-axis.* It is assumed that the function $f(x, y, z, \tau, \sigma)$ is of class $C^{iv}$† with respect to each of its five arguments in a region $R(x, y, z, \tau, \sigma)$, and throughout the discussion the variables will be confined to this region. The curves considered will be represented in the parametric form $(C)$, where $\phi, \psi, \chi$ are functions of class $C^{iv}$ defining values $(x, y, z, \tau, \sigma)$ interior to the region $R$ for all values of $t$ in the interval $t_0 \leqq t \leqq t_1$. Since $\tau$ and $\sigma$ are to be definitely determinate for every value of $t$ in this interval, it is assumed that $x_t, y_t, z_t$ do not all vanish simultaneously, *i. e.*, that the curves under consideration are without singular points.

If the integral $I$ is taken along a curve of the family

$$x=\phi(t, a), \quad y=\psi(t, a), \quad z=\chi(t, a), \tag{$C_a$}$$

where $\phi(t, a)$, $\psi(t, a)$, $\chi(t, a)$ are of class $C^{iv}$ in the region

$$t_0 \leqq t \leqq t_1, \quad |a| \leqq k > 0,$$

and for $a=0$ define the curve $C$, then the value of $I$ becomes a function $I(a)$, which for $a=0$ reduces to the value of $I$ taken along $C$. Hence the derivative of $I$ with respect to $a$ must vanish for $a=0$.

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*Bliss has developed the theory for the integral $\int f(x, y, \tau)\sqrt{x_t^2+y_t^2}\,dt$. See "A New Form of the Simplest Problem of the Calculus of Variations," *Transactions of the American Mathematical Society*, Vol. VIII (1907), p. 405.

†For definitions of terms and notations used in this paper see Bolza, "Vorlesungen über Variationsrechnung."

This derivative is found to be

$$\frac{dI}{da}=\int_{t_0}^{t_1}\Big(f_x x_a+f_y y_a+f_z z_a+f_\tau \tau_a+f_\sigma \sigma_a + f\frac{x_t x_{ta}+y_t y_{ta}+z_t z_{ta}}{x_t^2+y_t^2+z_t^2}\Big)\sqrt{x_t^2+y_t^2+z_t^2}\,dt. \tag{3}$$

If the values of $\tau_a, \sigma_a$ are obtained from (2) and substituted in (3), after some reductions

$$\frac{dI}{da}=\int_{s_0}^{s_1}(f_x x_a+f_y y_a+f_z z_a+p x_{sa}+q y_{sa}+r z_{sa})\,ds, \tag{4}$$

where

$$\left.\begin{aligned} p&=f\cos\sigma\cos\tau-f_\tau\frac{\sin\tau}{\cos\sigma}-f_\sigma\sin\sigma\cos\tau,\\ q&=f\cos\sigma\sin\tau+f_\tau\frac{\cos\tau}{\cos\sigma}-f_\sigma\sin\sigma\sin\tau,\\ r&=f\sin\sigma+f_\sigma\cos\sigma, \end{aligned}\right\} \tag{5}$$

and where $t$ has been replaced by $s$, the Euclidean length of arc.

The usual integration by parts gives

$$\frac{dI}{da}=\int_{s_0}^{s_1}(P x_a+Q y_a+R z_a)\,ds+[p x_a+q y_a+r z_a]_{s_0}^{s_1}, \tag{6}$$

in which

$$P=f_x-\frac{dp}{ds},\quad Q=f_y-\frac{dq}{ds},\quad R=f_z-\frac{dr}{dz}. \tag{7}$$

It follows at once that *the Euler equations are*

$$P=0,\quad Q=0,\quad R=0. \tag{8}$$

*These equations are not independent, however, for the relation*

$$P\cos\sigma\cos\tau+Q\cos\sigma\sin\tau+R\sin\sigma\equiv 0 \tag{9}$$

*is readily verified.*

When the differentiation indicated in (7) is carried out, equations (8) become

$$a_i\frac{d\tau}{ds}+b_i\frac{d\sigma}{ds}+c_i=0,\qquad i=1,2,3, \tag{10}$$

where the *a's*, *b's* and *c's* are defined as follows:

$$a_1=f\cos\sigma\sin\tau+f_\tau\sin\sigma\tan\sigma\cos\tau-f_\sigma\sin\sigma\sin\tau+f_{\tau\tau}\frac{\sin\tau}{\cos\sigma}+f_{\tau\sigma}\sin\sigma\cos\tau,$$

$$b_1=f\sin\sigma\cos\tau+f_\tau\frac{\sin\sigma\sin\tau}{\cos^2\sigma}+f_{\tau\sigma}\frac{\sin\tau}{\cos\sigma}+f_{\sigma\sigma}\sin\sigma\cos\tau,$$

$$\begin{aligned} c_1=&f_x(1-\cos^2\sigma\cos^2\tau)-f_y\cos^2\sigma\sin\tau\cos\tau-f_z\sin\sigma\cos\sigma\cos\tau\\ &+f_{x\tau}\sin\tau\cos\tau+f_{y\tau}\sin^2\tau+f_{z\tau}\tan\sigma\sin\tau\\ &+f_{x\sigma}\sin\sigma\cos\sigma+f_{y\sigma}\sin\sigma\cos\sigma\sin\tau\cos\tau+f_{z\sigma}\sin^2\sigma\cos\tau, \end{aligned}$$

$$a_2=-f\cos\sigma\cos\tau+f_\tau\sin\sigma\tan\sigma\sin\tau+f_\sigma\sin\sigma\cos\tau-f_{\tau\tau}\frac{\cos\tau}{\cos\sigma}+f_{\tau\sigma}\sin\sigma\sin\tau,$$

$$b_2=f\sin\sigma\sin\tau-f_\tau\frac{\sin\sigma\cos\tau}{\cos^2\sigma}-f_{\tau\sigma}\frac{\cos\tau}{\cos\sigma}+f_{\sigma\sigma}\sin\sigma\sin\tau,$$

$$\begin{aligned}c_2=&-f_x\cos^2\sigma\sin\tau\cos\tau+f_y(1-\cos^2\sigma\sin^2\tau)-f_z\sin\sigma\cos\sigma\sin\tau\\&-f_{x\tau}\cos^2\tau-f_{y\tau}\sin\tau\cos\tau-f_{z\tau}\tan\sigma\cos\tau\\&+f_{x\sigma}\sin\sigma\cos\sigma\sin\tau\cos\tau+f_{y\sigma}\sin\sigma\cos\sigma\sin^2\tau+f_{z\sigma}\sin^2\sigma\sin\tau,\end{aligned}$$

$$a_3=-f_\tau\sin\sigma-f_{\tau\sigma}\cos\sigma,$$

$$b_3=-f\cos\sigma-f_{\sigma\sigma}\cos\sigma,$$

$$\begin{aligned}c_3=&-f_x\sin\sigma\cos\sigma\cos\tau-f_y\sin\sigma\cos\sigma\sin\tau+f_z\cos^2\sigma\\&-f_{x\sigma}\cos^2\sigma\cos\tau-f_{y\sigma}\cos^2\sigma\sin\tau-f_{z\sigma}\sin\sigma\cos\sigma.\end{aligned}$$

The three equations (10) are linear and non-homogeneous in $\frac{d\tau}{ds}$, $\frac{d\sigma}{ds}$, and are consistent because of the identity (9). If any two-rowed determinant of the matrix

$$\left\|\begin{matrix}a_1 & b_1\\ a_2 & b_2\\ a_3 & b_3\end{matrix}\right\| \tag{11}$$

is different from zero, equations (10) can be solved for $\frac{d\tau}{ds}$, $\frac{d\sigma}{ds}$. The values of these determinants are found to be

$$\left|\begin{matrix}a_2 & b_2\\ a_3 & b_3\end{matrix}\right|=f_1\cos\tau,\quad \left|\begin{matrix}a_3 & b_3\\ a_1 & b_1\end{matrix}\right|=f_1\sin\tau,\quad \left|\begin{matrix}a_1 & b_1\\ a_2 & b_2\end{matrix}\right|=f_1\tan\sigma, \tag{12}$$

where

$$f_1=(f\cos^2\sigma-f_\sigma\sin\sigma\cos\sigma+f_{\tau\tau})(f+f_{\sigma\sigma})-(f_\tau\tan\sigma+f_{\tau\sigma})^2. \tag{13}$$

If $f_1\neq 0$, not all the determinants in (12) can be zero, and (10) can be solved, giving

$$\frac{d\tau}{ds}=A(x, y, z, \tau, \sigma),\quad \frac{d\sigma}{ds}=B(x, y, z, \tau, \sigma). \tag{14}$$

From (2),

$$\frac{dx}{ds}=\cos\sigma\cos\tau,\quad \frac{dy}{ds}=\cos\sigma\sin\tau,\quad \frac{dz}{ds}=\sin\sigma. \tag{15}$$

By reason of known existence theorems for differential equations, it may be stated that *in the region R, through the point* $x_0, y_0, z_0$, *and in the direction defined by* $\tau_0, \sigma_0$, *one and only one extremal can be drawn. The equations of these extremals are found by integrating* (14) *and* (15). *They have the form*

$$\left.\begin{aligned}x&=\phi(s, x_0, y_0, z_0, \tau_0, \sigma_0),\\ y&=\psi(s, x_0, y_0, z_0, \tau_0, \sigma_0),\\ z&=\chi(s, x_0, y_0, z_0, \tau_0, \sigma_0).\end{aligned}\right\} \tag{16}$$

*It may further be stated that the following initial conditions are satisfied:*

$$\left.\begin{aligned} &\phi(0, x_0, y_0, z_0, \tau_0, \sigma_0) = x_0, \quad \phi_s(0, x_0, y_0, z_0, \tau_0, \sigma_0) = \cos\sigma_0 \cos\tau_0, \\ &\psi(0, x_0, y_0, z_0, \tau_0, \sigma_0) = y_0, \quad \psi_s(0, x_0, y_0, z_0, \tau_0, \sigma_0) = \cos\sigma_0 \sin\tau_0, \\ &\chi(0, x_0, y_0, z_0, \tau_0, \sigma_0) = z_0, \quad \chi_s(0, x_0, y_0, z_0, \tau_0, \sigma_0) = \sin\sigma_0. \end{aligned}\right\} \quad (17)$$

## § 2. *The Jacobi Necessary Condition.*

If in equations (16) the constants $x_0, y_0, z_0$ are made equal to the coordinates of the point 0, and the constants $\tau_0, \sigma_0$ are replaced by functions $\tau(u, v), \sigma(u, v)$, which for $u=u_0, v=v_0$ reduce to the angles defining the direction of $C$ at the point 0, these equations take the form

$$x=\phi(s, u, v), \quad y=\psi(s, u, v), \quad z=\chi(s, u, v). \quad (18)$$

Equations (18) represent the extremal $C_{01}$ (*i. e.*, the extremal from the point 0 to the point 1) for $u=u_0, v=v_0$ and satisfy the identities

$$x_0=\phi(0, u, v), \quad y_0=\psi(0, u, v), \quad z_0=\chi(0, u, v), \quad (19)$$

which express the fact that all of the extremals pass through the point 0 for $s=0$.

The functional determinant of equations (18) is

$$\Delta(s, u, v) \equiv \begin{vmatrix} \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix},$$

and it can be shown in the usual way that if an extremal $C_{01}$ makes the integral $I$ either a maximum or a minimum, it can not have on it a point 2 at which the determinant $\Delta$ vanishes, *i. e.*, a point conjugate to the initial point 0, provided that at least one of the three-rowed determinants of the matrix

$$\left\| \begin{matrix} \Delta_s & \Delta_u & \Delta_v \\ \phi_s & \phi_u & \phi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{matrix} \right\|$$

is different from zero at the point 2.

## § 3. *The Hilbert Invariant Integral, the Weierstrass E-function, and the Necessary Condition of Weierstrass.*

An extremal arc $C_{01}$ that does not have on it a point 2 conjugate to the initial point 0, can be imbedded in a two-parameter family of extremals through a point 0′ in the neighborhood of 0 and in the order 0′01, whose determinant $\Delta$ is different from zero at every point of $C_{01}$. The family of extremals thus

forms a field about the extremal $C$, which is given by the parameter values $u=u_0$, $v=v_0$. Through each point of the region

$$|u-u_0|\leqq\delta, \quad |v-v_0|\leqq\delta, \tag{20}$$

provided $\delta$, which is a positive constant, is sufficiently small, there passes one, and but one, extremal of the set. Furthermore, $\delta$ can be so restricted that in the region (20) $\Delta$ is different from zero, and the equations

$$x=\phi(s, u, v), \quad y=\psi(s, u, v), \quad z=\chi(s, u, v) \tag{21}$$

have unique solutions

$$s=s(x, y, z), \quad u=u(x, y, z), \quad v=v(x, y, z) \tag{22}$$

of class $C'$. Consider a curve

$$x=\bar{x}(a), \quad y=\bar{y}(a), \quad z=\bar{z}(a) \tag{$\bar{C}$}$$

that lies entirely within the region (20). Equations (22) and ($\bar{C}$) determine $u$ and $v$ as functions of $a$. Substitution of these functions in (21) gives a one-parameter family of extremals

$$x=\phi(s, a), \quad y=\psi(s, a), \quad z=\chi(s, a), \tag{23}$$

all of which are of course members of the two-parameter family (21).

The value of the integral $I$ taken along an extremal of the set (23) from the point 0, where $s=0$ to the point 4, where the extremal intersects the curve $\bar{C}$ is obviously a function of $a$. By the method used in deriving the Euler equations it is found that

$$\frac{d}{da}I(C_{0'4})=f\frac{\partial s}{\partial a}\Big|^4+[p\phi_a+q\psi_a+r\chi_a]_{0'}^4, \tag{24}$$

$p$, $q$, $r$ being defined by (5). The terms in the brackets all vanish at the point 0', for all of the extremals pass through that point, and consequently $\phi_a$, $\psi_a$, $\chi_a$ are zero there. At the point 4,

$$\phi_s\frac{\partial s}{\partial a}+\phi_a=\bar{x}_a, \quad \psi_s\frac{\partial s}{\partial a}+\psi_a=\bar{y}_a, \quad \chi_s\frac{\partial s}{\partial a}+\chi_a=\bar{z}_a,$$

and, furthermore, $\phi$, $\psi$, $\chi$, $\tau$, $\sigma$ of (23) are equal, respectively, to $\phi$, $\psi$, $\chi$, $\tau$, $\sigma$ of (21) where the arguments $s$, $u$, $v$ of the latter are expressed as functions of $a$ by means of equations (22) and ($\bar{C}$). Since

$$p\phi_s+q\psi_s+r\chi_s=f,$$

equation (24) may be written

$$\frac{d}{da}I(C_{0'4})=[p\bar{x}_a+q\bar{y}_a+r\bar{z}_a]\,|^4,$$

the arguments $\phi(s, a)$, $\psi(s, a)$, $\chi(s, a)$, $\tau(s, a)$, $\sigma(s, a)$ of $f$, $f_\tau$, $f_\sigma$ in (24)

having been replaced by $\phi(s, u, v)$, $\psi(s, u, v)$, $\chi(s, u, v)$, $\tau(s, u, v)$, $\sigma(s, u, v)$ considered as functions of $a$.

Let $a_4$ be the value of $a$ that gives the extremal $C_{0'4}$ and $a_5$ the value of $a$ that gives the extremal $C_{0'5}$, where 5 as well as 4 is a point of the curve $\bar{C}$. Then

$$\begin{aligned} I(C_{0'5})-I(C_{0'4}) &= \int_{a_4}^{a_5}(p\bar{x}_a+q\bar{y}_a+r\bar{z}_a)\,da \\ &= \int_{a_4}^{a_5}\Big[\Big(f\cos\sigma\cos\tau-f_\tau\frac{\sin\tau}{\cos\sigma}-f_\sigma\sin\sigma\cos\tau\Big)\cos\bar{\sigma}\cos\bar{\tau} \\ &\quad+\Big(f\cos\sigma\sin\tau+f_\tau\frac{\cos\tau}{\cos\sigma}-f_\sigma\sin\sigma\sin\tau\Big)\cos\bar{\sigma}\sin\bar{\tau} \\ &\quad+(f\sin\sigma+f_\sigma\cos\sigma)\sin\bar{\sigma}\Big]\sqrt{\bar{x}_a^2+\bar{y}_a^2+\bar{z}_a^2}\,da \\ &= \int_{a_4}^{a_5}\Big\{f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)+\sin\sigma\sin\bar{\sigma}]+f_\tau\frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau) \\ &\quad-f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)-\cos\sigma\sin\bar{\sigma}]\Big\}\sqrt{\bar{x}_a^2+\bar{y}_a^2+\bar{z}_a^2}\,da, \end{aligned}$$

where $\bar{\tau}$, $\bar{\sigma}$ are the angles defining the direction of the curve $\bar{C}$.

*Since $I(C_{0'5})-I(C_{0'4})$ does not depend on the form of the curve $\bar{C}$, but merely on the points 4 and 5, it is seen that the integral*

$$\begin{aligned} I^* = \int\Big\{&f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)+\sin\sigma\sin\bar{\sigma}]+f_\tau\frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau) \\ &-f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)-\cos\sigma\sin\bar{\sigma}]\Big\}\sqrt{\bar{x}_a^2+\bar{y}_a^2+\bar{z}_a^2}\,da, \end{aligned}$$

*taken along a curve $\bar{C}$ in the field of extremals does not depend on the path of integration, but is a function of the end points only.* It is Hilbert's invariant integral and along an extremal reduces to the integral $I$, as is seen at once if $\bar{\tau}$, $\bar{\sigma}$ are set equal to $\tau$, $\sigma$, respectively.

The Weierstrass $E$-function is the function

$$\begin{aligned} e(x, y, z;\ \tau, \sigma;\ \bar{\tau}, \bar{\sigma}) = &f(x, y, z, \bar{\tau}, \bar{\sigma}) \\ &-f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)+\sin\sigma\sin\bar{\sigma}]-f_\tau\frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau) \\ &+f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)-\cos\sigma\sin\bar{\sigma}] \end{aligned}$$

occurring in the integrand of the difference

$$I(\bar{C})-I^*(\bar{C}) = \int e\sqrt{\bar{x}_a^2+\bar{y}_a^2+\bar{z}_a^2}\,da.$$

Weierstrass's necessary condition follows at once: *If the arc $C_{01}$ is a solution of the Euler equations that minimizes the integral $I$ with respect to*

*admissible curves $\overline{C}_{01}$ lying entirely within the field of extremals surrounding $C_{01}$, then the condition*

$$e(x, y, z;\ \tau, \sigma;\ \bar{\tau}, \bar{\sigma}) \geqq 0$$

*must be satisfied at every point of $C_{01}$ and for every direction $\bar{\tau}, \bar{\sigma}$. If $C_{01}$ maximizes $I$, $e$ must be $\leqq 0$.*

## § 4. *The Necessary Condition of Legendre.*

In order to deduce the necessary condition of Legendre, the connection between the $e$-function of the last section and the function $f_1$ defined in § 1 must be established. To do this, expand by Taylor's series with a remainder, each term of $e$ considered as a function of $\bar{\tau}$ and $\bar{\sigma}$. It is found that

$$\begin{aligned}
f(x, y, z, \bar{\tau}, \bar{\sigma}) &= f + (\bar{\tau}-\tau) f_\tau + (\bar{\sigma}-\sigma) f_\sigma \\
&\quad + \tfrac{1}{2}[(\bar{\tau}-\tau)^2 f_{\tau\tau} + 2(\bar{\tau}-\tau)(\bar{\sigma}-\sigma) f_{\tau\sigma} + (\bar{\sigma}-\sigma)^2 f_{\sigma\sigma}] + h_1, \\
f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau) + \sin\sigma\sin\bar{\sigma}] &= f + \tfrac{1}{2}[-(\bar{\tau}-\tau)^2 f\cos^2\sigma - (\bar{\sigma}-\sigma)^2 f] + h_2 f, \\
f_\tau \frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau) &= (\bar{\tau}-\tau) f_\tau - (\bar{\tau}-\tau)(\bar{\sigma}-\sigma) f_\tau \tan\sigma + h_3 f_\tau, \\
f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau) - \cos\sigma\sin\bar{\sigma}] &= -(\bar{\sigma}-\sigma) f_\sigma - \tfrac{1}{2}(\bar{\tau}-\tau)^2 f_\sigma \sin\sigma\cos\sigma + h_4 f_\sigma.
\end{aligned}$$

The remainder $h_1$ is homogeneous of degree 3 in $\bar{\tau}-\tau$, $\bar{\sigma}-\sigma$, and contains third partial derivatives of $f$ with mean value arguments $\tau^*$, $\sigma^*$. Since it has been assumed that $f$ is of class $C^{iv}$ with respect to all of its arguments, these partial derivatives are finite, and $h_1$ vanishes as $\bar{\tau}-\tau$, $\bar{\sigma}-\sigma$ both approach zero. The expressions $h_2$, $h_3$, $h_4$ are also homogeneous of degree 3 in $\bar{\tau}-\tau$, $\bar{\sigma}-\sigma$, and, consequently, converge to zero with these two quantities. Thus,

$$\begin{aligned}
e(x, y, z;\ \tau, \sigma;\ \bar{\tau}, \bar{\sigma}) &= \tfrac{1}{2}(\bar{\tau}-\tau)^2 (f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau}) \\
&\quad + (\bar{\tau}-\tau)(\bar{\sigma}-\sigma)(f_\tau\tan\sigma + f_{\tau\sigma}) + \tfrac{1}{2}(\bar{\sigma}-\sigma)^2 (f + f_{\sigma\sigma}) + h,
\end{aligned}$$

where $h$ is a finite expression that is homogeneous of degree 3 in $\bar{\tau}-\tau$, $\bar{\sigma}-\sigma$.

Consider now the quadratic form

$$K \equiv \xi^2 (f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau}) + 2\xi\eta(f_\tau\tan\sigma + f_{\tau\sigma}) + \eta^2(f + f_{\sigma\sigma})$$

in the real variables $\xi$, $\eta$. If the arc $C_{01}$ furnishes a minimum (either strong or weak) for the integral $I$, then $K$ must be $\geqq 0$ at all points of the arc $C_{01}$ and for all values of $\xi$ and $\eta$.

For if $\xi$, $\eta$ were a pair of values making $K$ negative at some point of $C_{01}$, $\bar{\tau}$ and $\bar{\sigma}$ could be chosen so that $\bar{\tau}-\tau = \varepsilon\xi$, $\bar{\sigma}-\sigma = \varepsilon\eta$, and for sufficiently small values of $\varepsilon$ the $e$-function would be negative. But it was seen in the preceding

section that $e$ must be $\geqq 0$ for a minimum. Similarly, for a maximum, $K$ must be $\leqq 0$.

*A necessary and sufficient condition that K be definite is that*

$$(f_\tau \tan\sigma + f_{\tau\sigma})^2 - (f\cos^2\sigma - f_\sigma \sin\sigma\cos\sigma + f_{\tau\tau})(f + f_{\sigma\sigma}) < 0,$$

*i. e., that* $f_1 > 0$. *Therefore, for a so-called regular problem (one in which* $f_1 \neq 0$*),* $f_1$ *must be positive if the extremal* $C_{01}$ *furnishes either a minimum or a maximum for the integral I.*

The relations

$$\left.\begin{aligned} &(f+f_{\sigma\sigma})K = f_1\xi^2 + [(f_\tau\tan\sigma + f_{\tau\sigma})\xi + (f+f_{\sigma\sigma})\eta]^2, \\ &(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})K \\ &\quad = [(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})\xi + (f_\tau\tan\sigma + f_{\tau\sigma})\eta]^2 + f_1\eta^2 \end{aligned}\right\} \quad (25)$$

are easily verified. If it is assumed that $f_1 > 0$, it follows that

$$K,\ f + f_{\sigma\sigma},\ f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau}$$

must all have the same sign. But if $f_1$ is positive, neither $f + f_{\sigma\sigma}$ nor $f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau}$ can be equal to zero. Consequently, they must be greater than zero for a minimum and less than zero for a maximum.

## § 5. *Sufficient Conditions for a Weak Extremum.*

The $e$-function can be expressed in forms similar to (25) of the preceding section:

$$2(f+f_{\sigma\sigma})e = f_1(\bar\tau-\tau)^2 + [(f_\tau\tan\sigma + f_{\tau\sigma})(\bar\tau-\tau) + (f+f_{\sigma\sigma})(\bar\sigma-\sigma)]^2 + 2(f+f_{\sigma\sigma})h$$

$$2(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})e$$
$$= [(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})(\bar\tau-\tau) + (f_\tau\tan\sigma + f_{\tau\sigma})(\bar\sigma-\sigma)]^2$$
$$+ f_1(\bar\sigma-\sigma)^2 + 2(f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})h$$

where $h$, as previously stated, is homogeneous of degree 3 in $\bar\tau-\tau$, $\bar\sigma-\sigma$.

For a regular problem, $f+f_{\sigma\sigma}$ and $f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau}$ are either both positive or else both negative. Assume them positive, *i. e.*, assume that Legendre's necessary condition for a minimum is fulfilled. Then, at least if $|\bar\tau-\tau|$ and $|\bar\sigma-\sigma|$ are less than some positive constant $\gamma$, the $e$-function is positive except when $\bar\tau=\tau$, $\bar\sigma=\sigma$, in which case it vanishes.

It follows that if along the extremal arc $C_{01}$ Jacobi's necessary condition for an extremum and Legendre's necessary condition for a minimum are satisfied, $C_{01}$ will furnish at least a weak minimum for the integral $I$. A similar theorem of course holds for a weak maximum.

## § 6. *The Transversality Condition.*

It can easily be proved that if one end point of the extremal $C$ is allowed to vary along a curve $\bar{C}$, then the condition

$$f[\cos\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)+\sin\sigma\sin\bar{\sigma}]+f_\tau\frac{\cos\bar{\sigma}}{\cos\sigma}\sin(\bar{\tau}-\tau)$$
$$-f_\sigma[\sin\sigma\cos\bar{\sigma}\cos(\bar{\tau}-\tau)-\cos\sigma\sin\bar{\sigma}]=0$$

must be satisfied at the point of intersection of $C$ and $\bar{C}$.

When $f=1$ this condition reduces to the condition for perpendicularity as is to be expected.

When $\bar{\sigma}=\sigma=0$ it reduces to the condition as given by Bliss for the problem in two dimensions.

## § 7. *A Generalized Notion of Angle.*

Consider the two-parameter family of extremals (18) that pass through the point 0. On each of the extremals take an extremal arc such that the value of the integral $I$ taken along the arc from the point 0 to the other end of the arc shall be equal to a constant $\rho$. The value of $s$ determining the end-point of this arc is defined by the equation

$$\rho=\int_0^s f(\phi,\psi,\chi,\tau,\sigma)\sqrt{\phi_s^2+\psi_s^2+\chi_s^2}\,ds,\quad \phi_s^2+\psi_s^2+\chi_s^2\equiv 1. \tag{26}$$

The locus of the terminal points of these arcs is a so-called transversal surface. Since $\rho$ may be considered as the generalized distance from the point 0 to this surface, the surface itself may be thought of as a generalized sphere whose center is the point 0 and whose radius is $\rho$.

Assuming that $f$ is greater than zero, $\dfrac{\partial\rho}{\partial s}=f$ is different from zero at the point 0, and (26) can be solved for $s$ in terms of $\rho$, $\tau_0$, $\sigma_0$, and the solution will have continuous first derivatives with respect to all three arguments in a properly chosen region. Since

$$\frac{\partial s}{\partial\rho}=\frac{1}{f}>0$$

in the vicinity of the point 0, the value of $s$ will increase from zero with $\rho$. The equations of the transversal surface are found by substituting in equations (16) the value of $s$ obtained from (26). The parameters of the surface are $\tau_0$ and $\sigma_0$. It is convenient to introduce here the following functions of the theory of surfaces:

$$E=x_{\tau_0}^2+y_{\tau_0}^2+z_{\tau_0}^2,\quad F=x_{\tau_0}x_{\sigma_0}+y_{\tau_0}y_{\sigma_0}+z_{\tau_0}z_{\sigma_0},\quad G=x_{\sigma_0}^2+y_{\sigma_0}^2+z_{\sigma_0}^2.$$

Consider any curve $\bar{C}$ on the transversal surface, and suppose that $\tau_0$ and $\sigma_0$ are functions of a single parameter $a$. Then the value of the integral $I$, taken over this curve from the point $A_0(a=a_0)$ to the point $A_1(a=a_1)$ is

$$\lambda=\int_{a_0}^{a_1} f(\bar{x}, \bar{y}, \bar{z}, \bar{\tau}, \bar{\sigma}) \sqrt{E\tau_0'^2+2F\tau_0'\sigma_0'+G\sigma_0'^2}\,da, \tag{27}$$

where $\bar{x}, \bar{y}, \bar{z}, \bar{\tau}, \bar{\sigma}$ refer to $\bar{C}$, and where accents denote differentiation with respect to $a$. The quantity $\lambda$ may be thought of as the generalized length of the curve $\bar{C}$ from the point $A_0$ to the point $A_1$.

Let the angle through which the extremal $OA_1$ has passed in moving to its position from the position $OA_0$ be defined as the limit, as $\rho$ approaches zero, of the ratio $\frac{\lambda}{\rho}$.

The process of calculating this limit is somewhat lengthy, but can be accomplished in a perfectly straightforward manner. By partial differentiation with respect to $\tau_0$ and $\sigma_0$, from (26) are obtained the two equations:

$$0=f(\phi, \psi, \chi, \tau, \sigma)\frac{\partial s}{\partial \tau_0}+\int_0^s \frac{\partial}{\partial \tau_0}[f(\phi, \psi, \chi, \tau, \sigma)\sqrt{\phi_s^2+\psi_s^2+\chi_s^2}]ds,$$

$$0=f(\phi, \psi, \chi, \tau, \sigma)\frac{\partial s}{\partial \sigma_0}+\int_0^s \frac{\partial}{\partial \sigma_0}[f(\phi, \psi, \chi, \tau, \sigma)\sqrt{\phi_s^2+\psi_s^2+\chi_s^2}]ds,$$

which reduce, when integrated by parts, to

$$\left.\begin{aligned} 0&=f\frac{\partial s}{\partial \tau_0}+p\phi_{\tau_0}+q\psi_{\tau_0}+r\chi_{\tau_0},\\ 0&=f\frac{\partial s}{\partial \sigma_0}+p\phi_{\sigma_0}+q\psi_{\sigma_0}+r\chi_{\sigma_0},\end{aligned}\right\} \tag{28}$$

Now

$$x_{\tau_0}=\phi_s\frac{\partial s}{\partial \tau_0}+\phi_{\tau_0}=\cos\sigma\cos\tau\frac{\partial s}{\partial \tau_0}+\phi_{\tau_0},$$

and there are similar expressions for $y_{\tau_0}, z_{\tau_0}$. By using the expression for $\frac{\partial s}{\partial \tau_0}$ obtained from (28), and making some reductions, the values of $x_{\tau_0}, y_{\tau_0}, z_{\tau_0}$ are found to be

$$\begin{aligned} x_{\tau_0}=&\frac{1}{f}[\phi_{\tau_0}(f-f\cos^2\sigma\cos^2\tau+f_\tau\sin\tau\cos\tau+f_\sigma\sin\sigma\cos\sigma\cos^2\tau)\\ &+\psi_{\tau_0}(-f\cos^2\sigma\sin\tau\cos\tau-f_\tau\cos^2\tau+f_\sigma\sin\sigma\cos\sigma\sin\tau\cos\tau)\\ &+\chi_{\tau_0}(-f\sin\sigma\cos\sigma\cos\tau-f_\sigma\cos^2\sigma\cos\tau)]\\ =&\frac{1}{f}[q(\phi_{\tau_0}\cos\sigma\sin\tau-\psi_{\tau_0}\cos\sigma\cos\tau)-r(\chi_{\tau_0}\cos\sigma\cos\tau-\phi_{\tau_0}\sin\sigma)], \end{aligned}$$

$$y_{\tau_0}=\frac{1}{f}\,[\phi_{\tau_0}(-f\cos^2\sigma\sin^2\tau+f_\tau\sin^2\tau+f_\sigma\sin\sigma\cos\sigma\sin\tau\cos\tau)$$
$$+\psi_{\tau_0}(f-f\cos^2\sigma\sin^2\tau-f_\tau\sin\tau\cos\tau+f_\sigma\sin\sigma\cos\sigma\sin^2\tau)$$
$$+\chi_{\tau_0}(-f\sin\sigma\cos\sigma\sin\tau-f_\sigma\cos^2\sigma\sin\tau)\,]$$
$$=\frac{1}{f}\,[r(\psi_{\tau_0}\sin\sigma-\chi_{\tau_0}\cos\sigma\sin\tau)-p(\phi_{\tau_0}\cos\sigma\sin\tau-\psi_{\tau_0}\cos\sigma\cos\tau)\,],$$

$$z_{\tau_0}=\frac{1}{f}\,[\phi_{\tau_0}(-f\sin\sigma\cos\sigma\cos\tau+f_\tau\tan\sigma\sin\tau+f_\sigma\sin^2\sigma\cos\tau)$$
$$+\psi_{\tau_0}(-f\sin\sigma\cos\sigma\sin\tau-f_\tau\tan\sigma\cos\tau+f_\sigma\sin^2\sigma\sin\tau)$$
$$+\chi_{\tau_0}(f-f\sin^2\sigma-f_\sigma\sin\sigma\cos\sigma)\,]$$
$$=\frac{1}{f}\,[p(\chi_{\tau_0}\cos\sigma\cos\tau-\phi_{\tau_0}\sin\sigma)-q(\psi_{\tau_0}\sin\sigma-\chi_{\tau_0}\cos\sigma\sin\tau)\,].$$

The values of $x_{\sigma_0}, y_{\sigma_0}, z_{\sigma_0}$ are obtained by replacing the subscript $\tau_0$ by $\sigma_0$ in the right-hand members of the last three equations.

Some limits which are needed in finding the value of the limit of $\frac{\lambda}{\rho}$ are now to be calculated. Since

$$\frac{ds}{d\rho}=\frac{1}{f(\phi,\psi,\chi,\tau,\sigma)},$$

it follows that

$$\lim_{\rho=0}\frac{s}{\rho}=\frac{1}{f(x_0,y_0,z_0,\tau_0,\sigma_0)}. \tag{29}$$

By the law of the mean

$$\phi_{\tau_0}(s,\tau_0,\sigma_0)=\phi_{\tau_0}(0,\tau_0,\sigma_0)+s\phi_{s\tau_0}(\theta s,\tau_0,\sigma_0),\quad 0<\theta<1,$$
$$=s\phi_{s\tau_0}(\theta s,\tau_0,\sigma_0),$$

since from (17) $\phi_{\tau_0}(0,\tau_0,\sigma_0)$ is identically zero. Thus,

$$\lim_{\rho=0}\frac{1}{\rho}\phi_{\tau_0}(s,\tau_0,\sigma_0)=\lim_{\rho=0}\frac{1}{\rho}s\phi_{s\tau_0}(\theta s,\tau_0,\sigma_0)=-\frac{\cos\sigma_0\sin\tau_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)},$$

using (29) and (17). Similarly,

$$\lim_{\rho=0}\frac{1}{\rho}\psi_{\tau_0}(s,\tau_0,\sigma_0)=\frac{\cos\sigma_0\cos\tau_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)},$$
$$\lim_{\rho=0}\frac{1}{\rho}\chi_{\tau_0}(s,\tau_0,\sigma_0)=0,$$
$$\lim_{\rho=0}\frac{1}{\rho}\phi_{\sigma_0}(s,\tau_0,\sigma_0)=-\frac{\sin\sigma_0\cos\tau_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)},$$
$$\lim_{\rho=0}\frac{1}{\rho}\psi_{\sigma_0}(s,\tau_0,\sigma_0)=-\frac{\sin\sigma_0\sin\tau_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)},$$
$$\lim_{\rho=0}\frac{1}{\rho}\chi_{\sigma_0}(s,\tau_0,\sigma_0)=\frac{\cos\sigma_0}{f(x_0,y_0,z_0,\tau_0,\sigma_0)}.$$

From these limits may be computed

$$\lim_{\rho=0}\frac{1}{\rho}x_{\tau_0}=-\frac{1}{f^2}\,(q\cos^2\sigma_0+r\sin\sigma_0\cos\sigma_0\sin\tau_0)$$

$$=-\frac{1}{f^2}\cos\sigma_0(f\sin\tau_0+f_\tau\cos\tau_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}y_{\tau_0}=\frac{1}{f^2}\,(p\cos^2\sigma_0+r\sin\sigma_0\cos\sigma_0\cos\tau_0)$$

$$=\frac{1}{f^2}\cos\sigma_0(f\cos\tau_0-f_\tau\sin\tau_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}z_{\tau_0}=\frac{1}{f^2}\,(p\sin\sigma_0\cos\sigma_0\sin\tau_0-q\sin\sigma_0\cos\sigma_0\cos\tau_0)=-\frac{1}{f^2}f_\tau\sin\sigma_0,$$

$$\lim_{\rho=0}\frac{1}{\rho}x_{\sigma_0}=-\frac{1}{f^2}r\cos\tau_0=-\frac{1}{f^2}\cos\tau_0(f\sin\sigma_0+f_\sigma\cos\sigma_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}y_{\sigma_0}=-\frac{1}{f^2}r\sin\tau_0=-\frac{1}{f^2}\sin\tau_0(f\sin\sigma_0+f_\sigma\cos\sigma_0),$$

$$\lim_{\rho=0}\frac{1}{\rho}z_{\sigma_0}=\frac{1}{f^2}\,(p\cos\tau_0+q\sin\tau_0)=\frac{1}{f^2}\,(f\cos\sigma_0-f_\sigma\sin\sigma_0),$$

from which it follows that

$$\lim_{\rho=0}\frac{E}{\rho^2}=\frac{1}{f^4}\,(f^2\cos^2\sigma_0+f_\tau^2),\quad \lim_{\rho=0}\frac{F}{\rho^2}=\frac{1}{f^4}f_\tau f_\sigma,\quad \lim_{\rho=0}\frac{G}{\rho^2}=\frac{1}{f^4}\,(f^2+f_\sigma^2).$$

Going back to (27), *it is found that*

$$\lim_{\rho=0}\frac{\lambda}{\rho}=\int_{a_0}^{a_1}\frac{f(x_0,y_0,z_0,\bar\tau_0,\bar\sigma_0)}{f(x_0,y_0,z_0,\tau_0,\sigma_0)}\sqrt{(f^2\cos^2\sigma_0+f_\tau^2)\tau_0'^2+2f_\tau f_\sigma\tau_0'\sigma_0'+(f^2+f_\sigma^2)\sigma_0'^2}\,da, \quad (30)$$

*which is the generalized expression for angle.*

In the case of Euclidean geometry, $f=1$, the extremals are straight lines, and the transversal surface is the sphere

$$x=\rho\cos\sigma_0\cos\tau_0,\quad y=\rho\cos\sigma_0\sin\tau_0,\quad z=\rho\sin\sigma_0,$$

for which $E=\rho^2\cos^2\sigma_0$, $F=0$, $G=\rho^2$. Since $f_\tau$ and $f_\sigma$ both vanish, the expression (30) reduces to

$$\int_{a_0}^{a_1}\sqrt{\cos^2\sigma_0\tau_0'^2+\sigma_0'^2}\,da, \quad (31)$$

which is the ordinary definition of the angle through which the line $OA$ has moved in passing from the position $OA_0$ to the position $OA_1$, the point $A$ traversing any curve on the sphere.

A slightly less general definition of angle in Euclidean geometry is the value of (31) when the point $A$ is restricted to move along a great circle of

the sphere, *i. e.*, when the line moves from $OA_0$ to $OA_1$ keeping always in the same plane. It will be shown that a curve $\overline{C}$ on the transversal surface (or in fact on any surface) that minimizes the integral (27) with respect to neighboring curves on the surface, must satisfy Euler equations, which for $f=1$ reduce to the differential equations of geodesics. Since the geodesics of a sphere are great circles, the value of the integral in (30) taken over such a curve as $\overline{C}$, which may be called a generalized geodesic, obviously reduces for Euclidean geometry to the length of the circular arc $A_0A_1$ divided by the radius of the sphere, which is the definition of the plane angle that $OA_1$ makes with $OA_0$. For simplicity write

$$\lambda=\int_{a_0}^{a_1}\bar{f}\Gamma da,$$

where

$$\bar{f}=f(\bar{x},\bar{y},\bar{z},\bar{\tau},\bar{\sigma}),\quad \Gamma=\sqrt{E\tau_0'^2+2F\tau_0'\sigma_0'+G\sigma_0'^2}.$$

It is readily found that *the curve* $\overline{C}$ *mentioned above must satisfy the two equations*

$$\begin{aligned}\bar{x}_{\tau_0}&\left[\bar{f}_x\Gamma+\frac{d}{da}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\&+\bar{y}_{\tau_0}\left[\bar{f}_y\Gamma-\frac{d}{da}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]\\&+\bar{z}_{\tau_0}\left[\bar{f}_z\Gamma-\frac{d}{da}\bar{f}_\sigma\cos\bar{\sigma}\right]+\bar{f}\Gamma_{\tau_0}-\frac{d}{da}\bar{f}\Gamma_{\tau_0'}=0,\end{aligned}\tag{32}$$

$$\begin{aligned}\bar{x}_{\sigma_0}&\left[\bar{f}_x\Gamma+\frac{d}{da}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\&+\bar{y}_{\sigma_0}\left[\bar{f}_y\Gamma-\frac{d}{da}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]\\&+\bar{z}_{\sigma_0}\left[\bar{f}_z\Gamma-\frac{d}{da}\bar{f}_\sigma\cos\bar{\sigma}\right]+\bar{f}\Gamma_{\sigma_0}-\frac{d}{da}\bar{f}\Gamma_{\sigma_0'}=0,\end{aligned}\tag{33}$$

where $\bar{\tau}$ and $\bar{\sigma}$, the angles determining direction on the curve $\overline{C}$, are defined analytically by the equations

$$\cos\bar{\sigma}\cos\bar{\tau}=\frac{\bar{x}'}{\sqrt{\bar{x}'^2+\bar{y}'^2+\bar{z}'^2}}=\frac{\bar{x}'}{\Gamma},$$

$$\cos\bar{\sigma}\sin\bar{\tau}=\frac{\bar{y}'}{\Gamma},\quad \sin\bar{\sigma}=\frac{\bar{z}'}{\Gamma}.$$

When $f=1$ equations (32) and (33) reduce to

$$\Gamma_{\tau_0}-\frac{d}{da}\Gamma_{\tau_0'}=0,\quad \Gamma_{\sigma_0}-\frac{d}{da}\Gamma_{\sigma_0'}=0.$$

These are the differential equations of geodesics on the surface whose parameters are $\tau_0$, $\sigma_0$.

Multiply (32) by $\tau_0'$ and (33) by $\sigma_0'$ and add, remembering that

$$\bar{x}_{\tau_0}\tau_0'+\bar{x}_{\sigma_0}\sigma_0'=\bar{x}'.$$

The result is the single equation

$$\begin{aligned}&\bar{x}'\left[\bar{f}_x\Gamma+\frac{d}{da}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\&+\bar{y}'\left[\bar{f}_y\Gamma-\frac{d}{da}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]+\bar{z}'\left[\bar{f}_z\Gamma-\frac{d}{da}\bar{f}_\sigma\cos\bar{\sigma}\right]\\&+\tau_0'\Gamma_{\tau_0}+\sigma_0'\Gamma_{\sigma_0}-\tau_0'\frac{d}{da}\bar{f}\Gamma_{\tau_0'}-\sigma_0'\frac{d}{da}\bar{f}\Gamma_{\sigma_0'}=0.\end{aligned}\tag{34}$$

Since $\Gamma$ satisfies the Weierstrass homogeneity condition, the last four terms of (34) may be written

$$(1-\bar{f})(\tau_0'\Gamma_{\tau_0}+\sigma_0'\Gamma_{\sigma_0})-\Gamma\frac{d}{da}\bar{f}=0.$$

Dividing (34) by $\Gamma$, which is assumed to be different from zero, the equation becomes

$$\begin{aligned}&\cos\bar{\sigma}\cos\bar{\tau}\left[\bar{f}_x\Gamma+\frac{d}{da}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\&+\cos\bar{\sigma}\sin\bar{\tau}\left[\bar{f}_y\Gamma-\frac{d}{da}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]\\&+\sin\bar{\sigma}\left[\bar{f}_z\Gamma-\frac{d}{da}\bar{f}_\sigma\cos\bar{\sigma}\right]+\frac{1-\bar{f}}{\Gamma}(\tau_0'\Gamma_{\tau_0}+\sigma_0'\Gamma_{\sigma_0'})-\frac{d}{da}\bar{f}=0.\end{aligned}$$

If $a$ is replaced by $s$, the Euclidean length of arc, $\Gamma=1$, and the last equation reduces to

$$\begin{aligned}&\cos\bar{\sigma}\cos\bar{\tau}\left[\bar{f}_x+\frac{d}{ds}\left(\bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}}+\bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau}\right)\right]\\&+\cos\bar{\sigma}\sin\bar{\tau}\left[\bar{f}_y-\frac{d}{ds}\left(\bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}}-\bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau}\right)\right]\\&+\sin\bar{\sigma}\left[\bar{f}_z-\frac{d}{ds}\bar{f}_\sigma\cos\bar{\sigma}\right]-\frac{d}{ds}\bar{f}=0.\end{aligned}$$

## § 8. *A Generalization of the Definition of Solid Angle.*

The definition of the measure of a solid angle is the area cut out on a unit sphere by the cone which makes the solid angle (the vertex of the cone of course being at the center of the sphere), or what is the same thing, the ratio of the area cut out on any sphere to the square of the radius of that sphere. In the present section is given a generalized definition of solid angle, which for Euclidean geometry reduces to the ordinary definition.

If on the transversal surface considered in the preceding section there is a closed curve $\overline{C}$, the area $\overline{S}$ enclosed by $\overline{C}$ is

$$\overline{S}=\iint_S \sqrt{EG-F^2}\,d\tau_0 d\sigma_0,$$

where $S$ is the image, in the $\tau_0$, $\sigma_0$ plane of $\overline{S}$. A more general definition of of such area is obtained by setting

$$\overline{S}=\iint_S g(\tau_0, \sigma_0, \tau_0', \sigma_0')\sqrt{EG-F^2}\,d\tau_0 d\sigma_0.$$

Solid angle may then be defined by

$$\lim_{\rho=0}\frac{\overline{S}}{\rho^2}.$$

The values of

$$\lim_{\rho=0}\frac{E}{\rho^2},\quad \lim_{\rho=0}\frac{F}{\rho^2},\quad \lim_{\rho=0}\frac{G}{\rho^2}$$

have already been found; from these values can be obtained

$$\lim_{\rho=0}\frac{1}{\rho^4}(EG-F^2)=\frac{1}{f^8}(f^2\cos^2\sigma_0+f_\tau^2)(f^2+f_\sigma^2)-\frac{1}{f^8}f_\tau^2 f_\sigma^2,$$

from which it follows that

$$\lim_{\rho=0}\frac{\overline{S}}{\rho^2}=\iint_S \frac{g}{f^4}\sqrt{(f^2\cos^2\sigma_0+f_\tau^2)(f^2+f_\sigma^2)-f_\tau^2 f_\sigma^2}\,d\tau_0 d\sigma_0.$$

It is obvious that for Euclidean geometry, when $f=g=1$ and the transversal surface is a sphere, this last expression reduces to the usual definition of solid angle as given above.

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# *Derivation of the Complementary Theorem from the Riemann-Roch Theorem.*

By Samuel Beatty.

In his book* and subsequent memoirs† on the "Theory of the Algebraic Functions of a Complex Variable," Dr. J. C. Fields gives several proofs of the complementary theorem, of which theorem the well-known Riemann-Roch theorem is a particular case. The purpose of the present paper is not to give an independent proof of the complementary theorem, but rather to derive the same from the Riemann-Roch theorem.

As a preliminary we shall state the complementary theorem in its most general form. We shall suppose that the fundamental equation is

$$f(z, u) = u^n + u^{n-1} f_1(z) + \ldots + f_n(z) = 0, \tag{1}$$

in which each $f_s(z)$ is a rational function of $z$. The only limitation on $f(z, u)$ is that it may not possess a repeated factor. The $n$ expansions of $u$ in the vicinity of a value $z = a_k$ group themselves into $r_k$ cycles of orders

$$\nu_1^{(k)}, \quad \nu_2^{(k)}, \ldots, \quad \nu_{r_k}^{(k)},$$

respectively. The expansions of a cycle of order $\nu_s^{(k)}$ proceed according to ascending integral powers of

$$(z - a_k)^{1/\nu_s^{(k)}}.$$

The $n$ expansions of $u$ in the vicinity of the value $z = \infty$ group themselves into $r_\infty$ cycles of orders

$$\nu_1^{(\infty)}, \quad \nu_2^{(\infty)}, \ldots, \quad \nu_{r_\infty}^{(\infty)},$$

respectively. The expansions of a cycle of order $\nu_s^{(\infty)}$ proceed according to ascending integral powers of

$$\left(\frac{1}{z}\right)^{1/\nu_s^{(\infty)}}.$$

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*Mayer and Muller, "Theory of the Algebraic Functions of a Complex Variable," Berlin (1916).

†One of the most recent and complete bears the title "On the Foundations of the Theory of Algebraic Functions of one Variable." *Phil. Trans. of the Royal Soc. of London*, Series A, Vol. CCXII, pp. 339–373.

Let us imagine we have a set of numbers, one to a cycle, in connection with all values of $z$. The nature of these numbers we require to be such that the number relating to a cycle of order $\nu_s^{(k)}$ in connection with a value $z=a_k$ is a multiple of $1/\nu_s^{(k)}$, while the number relating to a cycle of order $\nu_s^{(\infty)}$ in connection with the value $z=\infty$ is a multiple of $1/\nu_s^{(\infty)}$; moreover, only a finite number of such numbers may be different from zero. These numbers we agree to denote by $\tau_s^{(k)}$, $\tau_s^{(\infty)}$, respectively, and the set by $(\tau)$. Such a set $(\tau)$ is called a basis and the numbers $\tau_s^{(k)}$, $\tau_s^{(\infty)}$ are called elements of the basis. Those elements of $(\tau)$ relative to finite values of $z$ and to the value $z=\infty$ constitute the partial bases $(\tau)'$ and $(\tau)^\infty$, respectively. The order of coincidence of $G(z, u)$, a rational function of $(z, u)$, relative to a cycle of order $\nu_s^{(k)}$ in connection with a value $z=a_k$ is defined to be the least number appearing as a power of the element of expansion $z-a_k$ in the expression obtained on replacing $u$ in $G(z, u)$ by any of the expansions of $u$ relative to such cycle. The order of coincidence of $G(z, u)$ relative to a cycle in connection with the value $z=\infty$ is defined in a corresponding manner, where, of course, the element of expansion is $1/z$. Suppose now that $G(z, u)$ is a specific rational function of $(z, u)$ not identically zero for any of the branches. Then its orders of coincidence form a basis $(\gamma)$. The partial bases

$$\left(-\tau+\gamma-1+\frac{1}{\nu}\right)', \quad \left(-\tau+\gamma+1+\frac{1}{\nu}\right)^\infty$$

constitute a basis $(\bar{\tau})$ complementary adjoint to the basis $(\tau)$ to the level furnished by the function $G(z, u)$. A rational function of $(z, u)$ is built on a basis $(\tau)$ if each of its orders of coincidence is equal to or greater than the corresponding element of the basis $(\tau)$. Denoting the general rational function of $(z, u)$ built on a basis $(\tau)$ by $R_\tau(z, u)$ and the number of arbitrary constants appearing therein by $N_\tau$, the complementary theorem consists in the formula

$$N_\tau+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}\tau_s^{(k)}\nu_s^{(k)}=N_{\bar{\tau}}+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}\bar{\tau}_s^{(k)}\nu_s^{(k)}. \tag{2}$$

Moreover, for the purpose of the present paper this formula will be regarded as affording a statement of the Riemann-Roch theorem, where for such purpose, however, equation (1) must be irreducible, the specific rational function of $(z, u)$ used as a level in determining a basis $(\bar{\tau})$ complementary adjoint to $(\tau)$ must be $f'_u(z, u)$ and all the elements in one of the bases $(\tau)$, $(\bar{\tau})$ must be zero or negative. This statement of the Riemann-Roch theorem does not differ essentially from the usual one.

We now propose to derive the complementary theorem from the Riemann-Roch theorem. For the present we shall suppose that equation (1) is irredu-

cible and that $(\tau)$, $(\bar{\tau})$ are complementary adjoint to the level furnished by the function $f'_u(z,u)$. It may happen that not both $N_\tau$, $N_{\bar{\tau}}$ are zero; in that case we suppose that $N_\tau$ is different from zero. Then $R_\tau(z,u)$ does not vanish identically, and, consequently, particular values can be ascribed to the $N_\tau$ arbitrary constants appearing therein, in such a way as to insure that the resulting specific function $R(z,u)$ is not zero identically. But the orders of coincidence of $R(z,u)$ form a basis $(\rho)$. The bases $(\tau-\rho)$, $(\bar{\tau}+\rho)$ are complementary adjoint to the level furnished by the function $f'_u(z,u)$ and all the elements in the former of these are zero or negative. The Riemann-Roch theorem is, therefore, applicable and hence we may write

$$N_{\tau-\rho}+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}(\tau_s^{(k)}-\rho_s^{(k)})\nu_s^{(k)}=N_{\bar{\tau}+\rho}+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}(\bar{\tau}_s^{(k)}+\rho_s^{(k)})\nu_s^{(k)}. \tag{3}$$

The general rational functions of $(z,u)$ built on the bases $(\tau-\rho)$, $(\bar{\tau}+\rho)$ are evidently $R_\tau(z,u)/R(z,u)$, $R_{\bar{\tau}}(z,u)R(z,u)$, respectively, and consequently,

$$N_{\tau-\rho}=N_\tau, \quad N_{\bar{\tau}+\rho}=N_{\bar{\tau}}. \tag{4}$$

Moreover, the elements of the basis $(\rho)$ are connected by the relation

$$\sum_k\sum_{s=1}^{r_k}\rho_s^{(k)}\nu_s^{(k)}=0. \tag{5}$$

On utilizing formulae (4), (5) to effect a simplification of formula (3), the result is the complementary formula (2).

Still supposing that equation (1) is irreducible and that $(\tau)$, $(\bar{\tau})$ are complementary adjoint to the level furnished by the function $f'_u(z,u)$ it remains to treat the case in which $N_\tau$, $N_{\bar{\tau}}$ are both zero. Suppose that $(t)$ is a basis each element of which is equal to or less than the corresponding element of the basis $(\tau)$ and suppose, moreover, that $N_t$ is different from zero. Denote by $(\bar{t})$ the basis complementary adjoint to $(t)$ to the level furnished by the function $f'_u(z,u)$. We may, therefore, utilize the result of the previous paragraph in its application to the bases $(t)$, $(\bar{t})$ to arrive at the formula

$$N_t+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}t_s^{(k)}\nu_s^{(k)}=N_{\bar{t}}+\tfrac{1}{2}\sum_k\sum_{s=1}^{r_k}\bar{t}_s^{(k)}\nu_s^{(k)}. \tag{6}$$

But since in all cases

$$t_s^{(k)}+\bar{t}_s^{(k)}=\tau_s^{(k)}+\bar{\tau}_s^{(k)}, \tag{7}$$

and because each element $t_s^{(k)}$ is equal to or less than the corresponding element $\tau_s^{(k)}$, it follows that each element $\bar{t}_s^{(k)}$ is equal to or greater than the corresponding element $\bar{\tau}_s^{(k)}$ and, consequently,

$$N_{\bar{t}}=0. \tag{8}$$

By utilizing formulae (7), (8) we can give to formula (6) the form

$$\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)} - \overline{\tau}_s^{(k)})\nu_s^{(k)} = \sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)} - t_s^{(k)})\nu_s^{(k)} - N_t. \tag{9}$$

Now, the number of conditions to be applied to $R_t(z, u)$ to reduce it to $R_\tau(z, u)$ is certainly not more than

$$\sum_k \sum_{s=1}^{r_k} (\tau_s^{(k)} - t_s^{(k)})\nu_s^{(k)}$$

Since, however, the number of these conditions is $N_t$, it follows that the right side of (9) is not negative. That is,

$$\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)}\nu_s^{(k)} - \tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \overline{\tau}_s^{(k)}\nu_s^{(k)} \tag{10}$$

is not negative. By interchanging the rôles of the bases $(\tau)$, $(\overline{\tau})$, it follows in a similar manner that the expression in (10) is not positive. Therefore,

$$\tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \tau_s^{(k)}\nu_s^{(k)} = \tfrac{1}{2}\sum_k \sum_{s=1}^{r_k} \overline{\tau}_s^{(k)}\nu_s^{(k)}. \tag{11}$$

This is, however, the complementary formula for the case under discussion, since $N_\tau$, $N_{\overline{\tau}}$ are both zero. This completes the derivation of the complementary theorem from the Riemann-Roch theorem for the case in which equation (1) is irreducible, and in which $(\overline{\tau})$ is complementary adjoint to $(\tau)$ to the level furnished by the function $f'_u(z, u)$.

We shall denote by $(\tau) - \frac{1}{\nu}$ a basis, the elements of which with a single exception are all equal to the corresponding elements of a basis $(\tau)$ and for the excepted cycle of order $\nu$ the element is $\frac{1}{\nu}$ less than the corresponding element of $(\tau)$. Substituting in the above sentence for the word less the word greater we have the definition of a basis denoted by $(\tau) + \frac{1}{\nu}$. If $(\tau)$, $(\overline{\tau})$ are complementary adjoint to the level furnished by a specific rational function of $(z, u)$ not identically zero for any of the branches, it is evident that $(\tau) - \frac{1}{\nu}$, $(\overline{\tau}) + \frac{1}{\nu}$ are complementary adjoint to the same level. If equation (1) is irreducible, and if the specific rational function of $(z, u)$ used as level in determining a basis $(\overline{\tau})$ complementary adjoint to a given basis $(\tau)$ is $f'_u(z, u)$, then from the complementary formulae as already established relative to the

complementary bases $(\tau)$, $(\bar{\tau})$ and $(\tau)-\frac{1}{\nu}$, $(\bar{\tau})+\frac{1}{\nu}$, it readily follows that

$$(N_{\tau-\frac{1}{\nu}}-N_{\tau})+(N_{\bar{\tau}}-N_{\bar{\tau}+\frac{1}{\nu}})=1. \tag{12}$$

This formula is equivalent to the statement that of the two functions $R_{\tau-\frac{1}{\nu}}(z,u)$, $R_{\bar{\tau}}(z,u)$ one and only one possesses relative to the single cycle of order $\nu$ the precise order of the basis on which it is built.

Still supposing that equation (1) is irreducible, if any specific rational function of $(z,u)$ not identically zero is used as level in determining a basis $(\bar{\tau})$ complementary adjoint to a given basis $(\tau)$ it is plain that formula (12) holds good, since the quotient of $R_{\bar{\tau}}(z,u)$ by the specific rational function used as level in determining $(\bar{\tau})$, is the same whatever be the function employed as level, or in other words $N_{\bar{\tau}}$ is the same for all such bases $(\bar{\tau})$.

Finally, we shall prove that formula (12) is true, should equation (1) happen to be reducible. In that case suppose that $(\tau)$, $(\bar{\tau})$ are complementary adjoint to the level furnished by a specific rational function $G(z,u)$ not identically zero for any of the branches. Now,

$$(N_{\tau-\frac{1}{\nu}}-N_{\tau})+(N_{\bar{\tau}}-N_{\bar{\tau}+\frac{1}{\nu}}) \tag{13}$$

must possess one of the values 0, 1 or 2. Suppose that the single cycle of order $\nu$ for which $(\tau)-\frac{1}{\nu}$ differs from $(\tau)$ relates to the equation $f_i(z,u)=0$, which is any one of the irreducible equations comprised under equation (1). Denote by $R_{\tau}^{(i)}(z,u)$ the general rational function of $(z,u)$ conditioned by that part of a basis $(\tau)$ relating to the irreducible equation $f_i(z,u)=0$. We may then employ formula (12) as already arrived at, to conclude that one of the functions $R_{\tau-\frac{1}{\nu}}^{(i)}(z,u)$, $R_{\bar{\tau}}^{(i)}(z,u)$ possesses for the single cycle of order $\nu$ the precise order of the basis on which it is built. Writing $f(z,u)$ in the form $f_i(z,u)Q_i(z,u)$ and denoting the reduced form, relative to $f_i(z,u)=0$ of $R_{\tau}^{(i)}(z,u)/Q_i(z,u)$ by $H_{\tau}^{(i)}(z,u)$, it is plain that $H_{\tau-\frac{1}{\nu}}^{(i)}(z,u)Q_i(z,u)$, $H_{\bar{\tau}}^{(i)}(z,u)Q_i(z,u)$ are built on the bases $(\tau)-\frac{1}{\nu}$, $(\bar{\tau})$, respectively, relative to $f(z,u)=0$, and one of these functions possesses for the single cycle of order $\nu$ the precise order of the basis on which it is built. The same is, therefore, true for at least one of $R_{\tau-\frac{1}{\nu}}(z,u)$, $R_{\bar{\tau}}(z,u)$. In other words, expression (13) must equal 1 or 2. It is, however, not possible that it should possess the value 2 for then $R_{\tau-\frac{1}{\nu}}(z,u)R_{\bar{\tau}}(z,u)/G(z,u)$ would be a function with the sum of its

residues equal to $\nu$ times an arbitrary constant. Consequently, expression (13) equals 1 in all cases.

Just in passing, the writer wishes to remark that this result may be employed even more conveniently than the complementary theorem, of which it is a corollary, to demonstrate the existence of Abelian integrals of the third kind; also if equation (1) is irreducible, formula (12) may be employed to prove that the general $\phi$-function is either zero identically or possesses for any cycle the order of coincidence required by the $\phi$-basis; finally, if equation (1) is irreducible, it follows readily from formula (12) that a rational function of $(z, u)$ can be constructed with exactly one pole, when and only when the general $\phi$-function is identically zero.

We now proceed to derive the general complementary theorem from the generalized theorem contained in formula (12). Before doing so we shall state formula (12) in the form

$$(N_{\tau-\frac{1}{\nu}} - N_{\bar{\tau}+\frac{1}{\nu}}) = (N_\tau - N_{\bar{\tau}}) + 1. \tag{14}$$

The bases $(\tau)$, $(\bar{\tau})$ may be written in the forms

$$(\bar{\tau}) = (\tau) - (h) + (g), \quad (\tau) = (\bar{\tau}) + (h) - (g), \tag{15}$$

in which $(h)$, $(g)$ are bases, all the elements of which are zero or positive. By repeated application of formula (13) to the proper complementary bases and by addition of the results, we get the formulae

$$\left.\begin{aligned} (N_{\tau-h} - N_{\bar{\tau}+h}) &= (N_\tau - N_{\bar{\tau}}) + \sum_k \sum_{s=1}^{r_k} h_s^{(k)} \nu_s^{(k)}, \\ (N_{\bar{\tau}-g} - N_{\tau+g}) &= (N_{\bar{\tau}} - N_\tau) + \sum_k \sum_{s=1}^{r_k} g_s^{(k)} \nu_s^{(k)}. \end{aligned}\right\} \tag{16}$$

The left sides of these two equations are, however, equal. Hence,

$$2(N_\tau - N_{\bar{\tau}}) = \sum_k \sum_{s=1}^{r_k} g_s^{(k)} \nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} h_s^{(k)} \nu_s^{(k)}. \tag{17}$$

Replacing the right side of equation (17) by its equivalent value

$$\sum_k \sum_{s=1}^{r_k} \bar{\tau}_s^{(k)} \nu_s^{(k)} - \sum_k \sum_{s=1}^{r_k} \tau_s^{(k)} \nu_s^{(k)},$$

and dividing by 2 throughout, the result is the complementary formula, arrived at under the most general conditions.

UNIVERSITY OF TORONTO, *December* 31, 1915.

# *On the Equivalence of Relations $K_{q_1 q_2 m}$.*

By E. W. Chittenden.

## *Introduction.*

T. H. Hildebrandt * has developed the theory of systems $(\Omega; K)$ where $K$ is a relation $K_{q_1 q_2 m}$.† In terms of a relation $K$ a relation $L$ (limit) is defined. This leads, through the conditioning of the $K$ relation, to a basis system $(\Omega; K^{1367})$ which is found to be an adequate substitute for the system $(\Omega; V)$ of Fréchet, where $V$ denotes a voisinage.‡ In a system $(\Omega; K^{1367})$§ the relation $K$ may be unsymmetric in the arguments $q_1, q_2$; and the identical sequence consisting of a single repeated element $q$ may not have $q$ as a limit. This constitutes a reduction in the hypothesis of Fréchet.

It is shown in the present paper (§ 6) that from any system $(\Omega; K)$ a system $(\Omega; \dot{K})$ may be defined such that $\dot{K}$ is symmetric and the identical sequence for $q$ always has $q$ for a limit. If $K^{167}$, every continuous function in the system $(\Omega; K)$ is continuous in the derived system $(\Omega; \dot{K})$. The systems $(\Omega; K)$, $(\Omega; \dot{K})$ are said to be equivalent with respect to continuity.

The form of equivalence holding between $K$ and $\dot{K}$ follows from the relation between $L$ and $\dot{L}$ derived from $K$ and $\dot{K}$, respectively. The general theory of equivalences of systems determined by limit relations is important, and is developed, together with some of its consequences, in §§ 1, 2. Two systems $\tilde{\Sigma}$, $\hat{\Sigma}$ containing a class $\Omega$ and defining limit in $\Omega$ may have one of three types of equivalence, denoted by $(A)$, $(B)$, $(C)$. For example $(\Omega; K^{167})$, $(\Omega; \dot{K}^{167})$ are equivalent $(B)$. Each of the three types of equivalence implies its predecessor.

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* "A Contribution to the Foundations of Fréchet's Calcul Fonctionnel," American Journal of Mathematics, Vol. XXXIV (1912), pp. 237–290.

† This is a relation $K_2$, introduced by E. H. Moore, "Introduction to a Form of General Analysis." *The New Haven Mathematical Colloquium*, New Haven (1910), p. 126.

‡ *Rendiconti del Circulo di Palermo*, Vol. XXII, pp. 1–64.

§ Eight properties of a $K$ relation are defined by Hildebrandt, *loc. cit.*, pp. 243–244. The presence of property $i$ is denoted by $K^i$. The notation and results of Hildebrandt's memoirs are presupposed in the present paper.

In § 4 necessary and sufficient conditions for equivalence $(B)$ or $(C)$ of systems $(\mathfrak{Q}; \tilde{K}^1)$, $(\mathfrak{Q}; \hat{K}^1)$ are obtained. These conditions are applied in § 5 to show that a relation $K^{167}$ (unsymmetric) is equivalent $(C)$ to a relation $\bar{K}^{125}$ (symmetric), and also in § 6 to show that a relation $K^{1367}$ is equivalent $(B)$ to a relation $\bar{K}^{12345}$, therefore equivalent $(B)$ to a voisinage. Extensions of theorems of Fréchet obtained by Hildebrandt, which involve the hypotheses $K^{1367}$, are immediate consequences of this theory of equivalence.*

1. Denote by $S \equiv \{q_n\}$ an infinite sequence of elements of a class $\mathfrak{Q}$. We shall understand by the notation $qLS$ that $q$ is a limit of $S$. If $S$ consists of a single repeated element $q$, $S$ is the identical sequence $Iq$ for $q$. We consider systems $\Sigma$ containing a class $\mathfrak{Q}$ relative to which, for every element $q$ and sequence $S$, it is determined, whether or not $qLS$, $L$ being defined in some manner in the system $\Sigma$.

$(A)$ *Two systems* $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent with respect to limiting element if the derived class* $\mathfrak{R}'$ *of any subclass* $\mathfrak{R}$ *of* $\mathfrak{Q}$ *is the same in* $\tilde{\Sigma}$ *as in* $\hat{\Sigma}$.

$(B)$ $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent with respect to continuity if for every sequence* $S$ *and element* $q$, *such that* $Iq$ *is not a subsequence of* $S$, $q$ *is a limit of* $S$ *in either system if it is a limit of* $S$ *in the other.*

$(C)$ $\tilde{\Sigma}$; $\hat{\Sigma}$ *are equivalent with respect to limit of a sequence if every limit in one system is a limit in the other.*

The significance of the notation; $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent $(B)$ [equivalent $(C)$], is evident. Equivalence $(B)$ implies equivalence $(A)$ and is in turn implied by equivalence $(C)$. While it is desired to call attention to the large body of theorems which equivalence $(A)$ will carry over from a system $\tilde{\Sigma}$ to a system $\hat{\Sigma}$, nevertheless the principal results of this paper relate to equivalences $(B)$ and $(C)$.

If $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent $(B)$ and the respective limit relations $\tilde{L}$, $\hat{L}$ are of the type $L^2$ † then $q\tilde{L}S$ implies $q\hat{L}S'$, if $S'$ is any subsequence of $S$ which does not contain $Iq$. A similar statement holds for $q\hat{L}S$.

Two sequences $S'$, $S''$ are complementary subsequences of $S$ if every element of $S$ not in $S'$ is in $S''$. A relation $L$ is a relation $L^7$ if $qLS'$, $qLS''$ imply $qLS$.

*If* $\tilde{\Sigma}$, $\hat{\Sigma}$ *are equivalent with respect to continuity and* $\tilde{L}^{2467}$, $\hat{L}^{2467}$; *then* $\tilde{L}$, $\hat{L}$ *are identical.*

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* For example, the theorems of Hildebrandt, *loc. cit.*, § 22, pp. 288–290.

† A relation $L$ is of type $L^2$ in case $qLS$ implies $qLS'$, where $S'$ is any subsequence of $S$. Properties 1–6 of a relation $L$ are defined by Hildebrandt, *loc. cit.*, pp. 281–282.

We must show that, given $q\tilde{L}S$, $q\hat{L}S$ follows. If $S$ does not contain $Iq$, $q\hat{L}S$ is implied by the definition of equivalence with respect to continuity. If, except for a finite set of elements, $S$ coincides with $Iq$, we have from $\tilde{L}^2$, $q\tilde{L}Iq$, and from $\hat{L}^6$, $q\hat{L}Iq$. Hence from $\hat{L}^4$, $q\hat{L}S$. Otherwise $S$ contains $Iq$ and an infinite complementary sequence $S'$ which does not contain $Iq$. Therefore $q\hat{L}Iq$, $q\hat{L}S'$; and it follows from $\hat{L}^7$ that $q\hat{L}S$. A repetition of the argument with $\tilde{L}$, $\hat{L}$ interchanged completes the proof.

*If $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent with respect to continuity and $\tilde{L}^2$, $\hat{L}^2$, every function* $\mu$ continuous on $\Omega$ in either system is continuous in the other.*

We have to show, for example, that if $\mu$ is continuous on $\Omega$ in $\tilde{\Sigma}$ and $q\hat{L}S$, $(S \equiv \{q_n\})$ then $L_n \mu_{q_n} = \mu_q$. If $S$ differs from $Iq$ only in a finite number of elements, the result is immediate. Otherwise $S$ will contain $S'$ complementary to $Iq$. From $\hat{L}^2$, $q\hat{L}S'$; and from equivalence $q\tilde{L}S'$. Therefore $L_n \mu_{q'_n} = \mu_q$, and we have immediately $L_n \mu_{q_n} = \mu_q$.

2. The following properties of classes are definable in terms of limiting element: compact, closed, interior, perfect, separable. Hence,

*If $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent (A) [with respect to limiting element] the properties compact, closed, etc., of classes have the same significance in one system as in the other.*

That is, for example, if $\mathfrak{N}$ is compact in $\tilde{\Sigma}$ it is compact in $\hat{\Sigma}$.

*If $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent (B) [with respect to continuity] and $\tilde{L}^2$, $\hat{L}^2$ then in addition to the preceding proposition every function continuous in one system is continuous in the other.*

*If $\tilde{\Sigma}$, $\hat{\Sigma}$ are equivalent (C) [with respect to limit of a sequence] and $\tilde{L}^2$, $\hat{L}^2$ we have the results above and the further result that $\tilde{L}$ and $\hat{L}$ are identical relations.*

3. In terms of a relation $K$ we define a relation $L$ (limit) as follows:† given a sequence $\{q_n\}$ and an element $q$ such that for every integer $m$ there exists an integer $n_m$ such that $n = \mathfrak{N}_m$ implies

$$K_{q_n q m},$$

$q$ is said to be a limit of the sequence $\{q_n\}$, and we write

$$q = \underset{n}{L} q_n.$$

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*The relation (B) between $\tilde{L}$ and $\hat{L}$ derives its name from the fact expressed in this proposition.

†Cf. Hildebrandt, *loc. cit.*, p. 249. This definition differs from that of Fréchet in terms of voisinage only in non-essentials.

The relation $L$ so defined is a relation $L^K$ (dependent on $K$), and is a relation $L^{23457}$.* Unless $K$ is conditioned *limit* is not necessarily unique. It is evident that from different relations $K$, distinct relations $L$ may be defined for a single class $\mathfrak{Q}$.

It will be noticed that in case the relation $K$ is unsymmetric, the relations $K_{q_nqm}$ and $K_{qq_nm}$ may have different significance, and that if we replace $K_{q_nqm}$ by $K_{qq_nm}$ in the definition above a different relation $L$ may result. A number of relations $L$ may be defined for $K$ unsymmetric which reduce to the same relation under the hypothesis of symmetry. The following example will serve to make clear the significance of these remarks. Let $\mathfrak{Q}$ denote the interval $0 \leqq q \leqq 1$. The relation $K_{q_1q_2m}$ holds if $q_1 \leqq q_2$ and $|q_1 - q_2| = \frac{1}{2^m}$. Then $q = \underset{n}{L} q_n$ implies that the elements $q_n$ all lie to the left of $q$, if $n$ is sufficiently large; *i. e.*, $q$ is a right-hand limit of the sequence $\{q_n\}$. The relation $K_{qq_nm}$ may hold for every $m$ and $n \geqq n_m$, but $q$ will not be a limit of $\{q_n\}$ in this sense.

If we denote by $\underline{K}$ a relation such that

$$K_{q_1q_2m} \text{ implies } \underline{K}_{q_2q_1m},$$

we call $\underline{K}$ the *conjugate* of $K$. The limit relation $\underline{L}$ will be the conjugate of the relation $L$ defined in terms of $K$. In the example above the relations $L$, $\underline{L}$ are distinct, since $\underline{K}_{q_1q_2m}$ implies $q_1 \leqq q_2$, and therefore $q = \underset{n}{\underline{L}} q_n$ implies $q$ is a left-hand limit.

4. Denote by $\phi(q, m)$ any integral-valued single-valued function of two arguments $q$, $m$ such that for every $q$

$$\underset{m}{L} \phi(q, m) = \infty.$$

In terms of these functions $\phi$ we state the theorem:

THEOREM I. *A necessary and sufficient condition that two systems* $(\mathfrak{Q}; \hat{K}^1)$,† $(\mathfrak{Q}; \tilde{K}^1)$ *be equivalent with respect to limit of a sequence is that there exist a function* $\phi$, *and for every element* $q_2$ *an* $m_2$, *such that for every* $m \geqq m_2$,

$$\hat{K}_{q_1q_2m} \text{ implies } \tilde{K}_{q_1q_2\phi(q_2, m)}, \tag{a}$$

$$\tilde{K}_{q_1q_2m} \text{ implies } \hat{K}_{q_1q_2\phi(q_2, m)}; \tag{b}$$

*furthermore, if in each case* $q_1$ *is supposed to be distinct from* $q_2$, *conditions* (a) *and* (b) *become necessary and sufficient for equivalence of* $(\mathfrak{Q}; \hat{K}^1)$ $(\mathfrak{Q}; \tilde{K}^1)$ *with respect to continuity.*

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*Cf. Hildebrandt, *loc. cit.*, p. 249. That $L^7$ is an immediate consequence of the definition of limit.

†A relation $K$ is a relation $K^1$ if $K_{q_1q_2m}$ implies $K_{q_1q_2m'}$, where $m'$ is any integer less than $m$.

A proof of the first part of the theorem follows. A slight modification of this proof serves to establish the second part of the theorem.

We will establish the sufficiency of the condition by showing that $\hat{L}$ and $\tilde{L}$ are identical relations. From the symmetrical nature of the situation it will be sufficient to show that if $q=\underset{n}{\hat{L}} q_n$ then $q=\underset{n}{\tilde{L}} q_n$.

By definition, from $q=\underset{n}{\hat{L}} q_n$ we have for every integer $m_0$ an integer $n_{m_0}$ such that $n \geqq n_{m_0}$ implies

$$\hat{K}_{q_n q m_0}.$$

Condition (a) gives for $m_0 \geqq m_2$ (dependent on $q$)

$$\tilde{K}_{\bar{q}_n q \phi(q, m_0)}.$$

Since $\underset{m}{L}\phi(q, m)=\infty$ we have for every $m$ an $m_0$ such that $\phi(q, m_0) \geqq m$. If we choose $n_m \geqq (n_m, n_{m_2})$ we have for every $n \geqq n_m$, because of $\tilde{K}^1$ and the preceding relation,

$$\tilde{K}_{q_n q m}.$$

Therefore $q=\underset{m}{\tilde{L}} q_n$, as was to be proved.

We complete the proof by showing that conditions (a) and (b) are necessary consequences of the definition of equivalence for systems $(\Omega; K)$. Denote by $\hat{\Omega}_{q_2 m}$ the class of all elements $q_1$ in the relation $\hat{K}_{q_1 q_2 m}$ but not in the relation $\hat{K}_{q_1 q_2 (m+1)}$. Denote by $\tilde{m}_{q_1}$ the greatest value of $m$ for which the relation $\tilde{K}_{q_1 q_2 m}$ holds, in case such greatest value exists.

We now define $\tilde{\phi}(q_2, m)$ as follows: If among the values $\tilde{m}_{q_1}$ for elements $q_1$ of $\hat{\Omega}_{q_2 m}$ there is a least, take $\tilde{\phi}(q_2, m)$ equal to this $\tilde{m}_{q_1}$. If there is no least $\tilde{m}_{q_1}$ there is either no $\tilde{m}_{q_1}$ or there is a $q_1$ such that $\tilde{m}_{q_1}<1$. If there is no $\tilde{m}_{q_1}$, take $\tilde{\phi}(q_2, m)=m$. In the remaining case $\tilde{\phi}(q_2, m)=\tilde{m}_{q_1}<1$. It is desired to prove that for every $q_2$, $\underset{m}{L}\tilde{\phi}(q_2, m)=\infty$. Suppose there exists $\overline{m}$ such that for every $m$, $\phi(q_2, m)<m$. Consider first the values of $m$ for which $\tilde{\phi}(q_2, m)=\tilde{m}_{q_{1m}}$ for some $q_{1m}$ in $\hat{\Omega}_{q_2 m}$. From $\hat{K}^1$ it follows that either every $m$ is of this type, or else there is a least $m$ for which there exists a $q_{1m}$ in $\hat{\Omega}_{q_2 m}$ which possesses an $\tilde{m}_{q_1}$. In the first case $q_2=\underset{m}{L} q_{1m}$. From the equivalence assumed in the hypothesis, $q_2=\underset{m}{L} q_{1m}$. This contradicts the assumption that all the $m_{q_{1m}}$ are less than $\overline{m}$. In the second case there exists an $m_0$ such that for every $m>m_0$, $\tilde{\phi}(q_2, m)=m$, and therefore $\underset{m}{L}\phi(q_2, m)=\infty$. The assumption $\tilde{\phi}(q_2, m)<\overline{m}$ is therefore contradicted in every case.

We define a number $\tilde{m}_2$ as follows: Either $\tilde{\phi}(q_2, m)=\tilde{m}_{q_1m}$ for every $m$, or there exists $m_0$ such that for $m'\geqq m_0$, $\tilde{\phi}(q_2, m')=m'$. In the first case take $\tilde{m}_2$ such that $\tilde{\phi}(q_2, \tilde{m})\geqq 1$; in the second case take $\tilde{m}_2=m_0$. Then it is only necessary to take account of $\tilde{K}^1$, and the definitions of $\tilde{m}_2$, $\tilde{m}_{q_1}$, $\tilde{\phi}(q_2, m)$ to see that for every $m\geqq m_2$ condition (a) is satisfied.

By means of a similar choice of $\hat{m}_2$, $\hat{\phi}(q_2, m)$ we may satisfy condition (b). Choosing $m_2\geqq(\tilde{m}_2, \hat{m}_2)$ and $\phi(q_2, m)$ equal to the lesser of $\tilde{\phi}(q_2, m)$, $\hat{\phi}(q_2, m)$ we have $m_2$ and $\phi$ (dependent on $q$) as required. This completes the proof of the first part of Theorem I.

The hypothesis of equivalence with respect to limit of sequence of $K^1$ and $\underline{K}^1$ (conjugate of $K$) therefore leads to the existence of the function $\phi$ of Theorem I. We show in § 5 that if $K^1$ and $\underline{K}^1$ are equivalent $(C)$ they are equivalent $(C)$ to a symmetric relation $\bar{K}^1$ derived from $K^1$. In view of the fact that $L$ and $\underline{L}$ may be distinct for unsymmetric relations $K$, while their identity implies equivalence $(C)$ to a symmetric $K$ it would seem desirable to confine attention to the symmetric relations in applications of $K$ relations to the study of classes and functions. Especially when the resulting simplification of the treatment is considered.

There may exist a function $\phi$, effective in Theorem I whose values are independent of $q_2$. We thus obtain a corollary to this theorem:

COROLLARY: *A sufficient condition that two systems* $(\Omega; \hat{K}^1)$ $(\Omega; \tilde{K}^1)$ *be equivalent* $(C)$ *is that there exists a function* $\phi$, *such that for every* $m$ *and* $q_1$, $q_2$

$$\hat{K}_{q_1q_2m} \text{ implies } \tilde{K}_{q_1q_2\phi(m)}, \tag{a}$$

$$\tilde{K}_{q_1q_2m} \text{ implies } \hat{K}_{q_1q_2\phi(m)}. \tag{b}$$

That these conditions are not necessary will be seen if we suppose $\Omega$ represents the open interval $0\leqq q\leqq 1$ and define $\hat{K}$, $\tilde{K}$ as follows:

For $m<1$, every pair of elements $q_1q_2$ is in the relations $\hat{K}_{q_1q_2m}$, $\tilde{K}_{q_1q_2m}$. For $m\geqq 1$, we take

$$\hat{K}_{q_1q_2m} \equiv |q_1-q_2|\leqq\frac{1}{2^m}.^*$$

If $(k-1)/k=k/(k+1)$ then

$$\tilde{K}_{q_1q_2m} \equiv |q_1-q_2|\leqq\frac{1}{(k+1)^m}.$$

From the definitions $\hat{K}$, $\tilde{K}$ are symmetric. They are easily seen to be equivalent $(C)$, but the hypothesis of existence of a function $\phi$, leads to a contradiction.

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* That is, the relation $K_{q_1q_2m}$ is defined by this inequality for every pair of elements $q_1$, $q_2$.

5. Given an unsymmetric relation $K$ we may define a symmetric relation $\overline{K}$ as follows:

A pair of elements $q_1, q_2$ are in the relation $\overline{K}_{q_1q_2m}$ if and only if they are in one of the relations $K_{q_1q_2m}, K_{q_2q_1m}$.

It is easy to see that if $K^i$ $(i=1, 2, 3, 4)$ then $\overline{K}^i$ also.

*Theorem* II. $K^{167}$ *implies* $\overline{K}^{125(678)}$.*

The properties (6), (7), (8) are each equivalent to 5 because of $\overline{K}^2$. We have to show that there exists a function $\phi$ such that

$$\overline{K}_{q_1q_2m}, \overline{K}_{q_2q_3m} \text{ imply } \overline{K}_{q_1q_3\phi(m)}.$$

Now $\overline{K}_{q_1q_2m}$, $\overline{K}_{q_2q_3m}$, by definition of $\overline{K}$, imply that one of the following four pairs of relations is holding:

$$K_{q_1q_2m}, \quad K_{q_2q_3m};$$
$$K_{q_1q_2m}, \quad K_{q_3q_2m};$$
$$K_{q_2q_1m}, \quad K_{q_2q_3m};$$
$$K_{q_2q_1m}, \quad K_{q_3q_2m}.$$

If we choose a function $\phi$ effective in each of the four instances 5, 6, 7, 8, as is possible when $K^1$, and recall that $K^{167}$ implies $K^{58}$,† we obtain

$$\overline{K}_{q_1q_3\phi(m)}.$$

This, because of the definition of $\overline{K}$, implies

$$\overline{K}_{q_1q_3\phi(m)}.$$

The following proposition is now evident:

*$K^{167}$ implies that there exists a function $\phi$ such that $\overline{K}_{q_1q_2m}$ implies $K_{q_1q_2\phi(m)}$ and $K_{q_2q_1\phi(m)}$.*

We have shown that from an unsymmetric relation $K^1$ a symmetric relation $\overline{K}^1$ may always be defined. The converse is not true. Suppose that for every $m$ and $q_1$, $q_2$ the relation $\overline{K}_{q_1q_2m}$ holds. $\overline{K}$ is symmetric. It is evident that a relation $K^1$ can be made unsymmetric only by adding new relations $K_{q_1q_2m}$ to those already established. But this is impossible in the case cited.

Assuming that for at least one pair $\bar{q}_1, \bar{q}_2$ there is an $m_0$ such that for $m \geqq m_0$ the relation $\overline{K}_{\bar{q}_1\bar{q}_2m_1}$ does not hold, we define $K$ unsymmetric by assuming that the relation $\overline{K}_{q_1q_2m}$ implies $K_{q_1q_2m}$ and in addition $K_{\bar{q}_1\bar{q}_2(m_0+1)}$, but not $K_{\bar{q}_2\bar{q}_1(m+1)}$. Then $\overline{K}^i$ implies $K^i$ where $i$ denotes any of the properties (1), (2), (3), (4), (5), (6), (7), (8).

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* For definitions of the properties 1–8 of a $K$ relation see Hildebrandt, *loc. cit.*, pp. 244–246.

† Hildebrandt, *loc. cit.*, p. 246.

In § 4 it was remarked that if $K^1$ and $\underline{K}^i$ (conjugate) were equivalent $(C)$ they would be together equivalent $(C)$ to a symmetric relation $K$. This relation is in fact $\overline{K}$. Since $K, \underline{K}$ are equivalent $(C)$ we have (Theorem I) for every $q_2$ an $m_2$ and a function $\phi$, such that for every $m \geqq m_2$

$$K_{q_1 q_2 m} \text{ implies } \underline{K}_{q_1 q_2 \phi(q_2, m)} \text{ which implies } K_{q_2 q_1 \phi(q_2, m)}, \tag{a}$$

$$\underline{K}_{q_1 q_2 m} \text{ implies } K_{q_1 q_2 \phi(q_2, m)} \text{ which implies } \underline{K}_{q_2 q_1 \phi(q_2, m)}. \tag{b}$$

To demonstrate the equivalence $(C)$ of $K$ and $\overline{K}$, we choose $m_2$ so that (a) and (b) are satisfied, and $\phi(q_2, m) = (\text{lesser of } m, \phi(q_2, m))$. Then the relations:

$$K_{q_1 q_2 m} \text{ implies } \overline{K}_{q_1 q_2 \phi(q_2, m)},$$
$$\overline{K}_{q_1 q_2 m} \text{ implies } K_{q_1 q_2 \phi(q_2, m)},$$

are easily seen to hold for all $m \geqq m_2$. The equivalence $(C)$ of $\overline{K}$ and $\underline{K}$ is similarly established.

THEOREM III. *Any system* $(\Omega; K^{167})$ *is equivalent to a system* $(\Omega; K^{125})$.

This theorem is a consequence of Theorems I and II. To demonstrate the application of Theorem I we must show that conditions (a), (b) are fulfilled. It is sufficient to remark that $\bar{\phi}_m \leqq (m, \phi_m)$, where $\phi$ is effective in the proposition of § 5, and $m_2 = 1$, are effective in this instance.

In view of this theorem and § 2, it follows that the theorems of Hildebrandt,* which involve the hypothesis $K^{167}$ are not more general than the corresponding theorems with hypothesis $K^{125}$, that is the hypothesis of an unsymmetrical $K$ does not in this instance lead to more general results.

6. From $K$ we obtain $\dot{K}^4$ as follows:

$$K_{q_1 q_2 m} \text{ implies } \dot{K}_{q_1 q_2 m}(m); \tag{a}$$

$$\text{for every } q \text{ and } m \text{ we have } \dot{K}_{qqm}. \tag{b}$$

Then $\dot{K}_{q_1 q_2 m}$ implies either $K_{q_1 q_2 m}$ or $q_1 = q_2$. It is easy to see that $K^i$ implies $\dot{K}^i$ where $i$ is any of the properties (1), (2), ...., (8). Furthermore, $K^1$ and $\dot{K}^1$ are equivalent $(B)$. In fact $\phi(q_1', m) = m$ is effective in conditions (a), (b) of Theorem I.

THEOREM IV. *Any system* $(\Omega; K^{1367})$ *is equivalent* $(B)$ *to a system* $(\Omega; K^{12345})$.

From Theorem III, $K^{167}$ is equivalent $(C)$ to $\overline{K}^{125}$. But $\overline{K}^{1235}$ is equivalent $(B)$ to $\dot{\overline{K}}^{12345}$. Hence $K^{1367}$ is equivalent $(B)$ to $\dot{\overline{K}}^{12345}$.

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* *Loc. cit.*, pp. 273–290.

Hildebrandt has shown* that the concepts $K^{12345}$ and voisinage ($V$) are equivalent. Therefore, we have:

THEOREM V. *Any system* $(\Omega; K^{1367})$ *is equivalent* ($B$) *to a system* $(\Omega; V)$ *of Fréchet, and furthermore, any system* $(\Omega; K^{13467})$ *is equivalent* ($C$) *to a system* $(\Omega; V)$.

As an application of this theorem we call attention to the theorem:†

*In a system* $(\Omega; K^{1367})$ *a necessary and sufficient condition that every continuous function on* $\mathfrak{R}$ *be bounded on* $\mathfrak{R}$ *is that* $\mathfrak{R}$ *be extremal,*

which may be obtained from Hahn's‡ extension of a theorem of Fréchet in the following manner. The condition is necessary in a system $(\Omega; V)$. A system $(\Omega; K^{1367})$ is equivalent ($B$) to a system $(\Omega; V)$. Therefore every function continuous on $\mathfrak{R}$ in $(\Omega; V)$ is continuous on $\mathfrak{R}$ in $(\Omega; K)$. If every function continuous on $\mathfrak{R}$ in $(\Omega; K)$ is bounded on $\mathfrak{R}$ then $\mathfrak{R}$ is extremal in $(\Omega; V)$. From § 2 $\mathfrak{R}$ is extremal in $(\Omega; K)$, which was to be proved. The condition is sufficient since $\mathfrak{R}$ extremal in a system $(\Omega; K)$ is extremal in any system equivalent ($B$) to $(\Omega; K)$, that is in $(\Omega; V)$.

URBANA, ILLINOIS, *October* 14, 1916.

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* *Loc. cit.*, p. 248.

† Cf. Hildebrandt, *loc. cit.*, pp. 288–290.

‡ Cf. *Monatshefte für Math. u. Physik*, Vol. XIX, p. 251 ff.

# *Some Properties of Certain Finite Algebras.**

By Edward Kircher.

The object of this paper is to study some of the properties of a finite algebra $\mathfrak{A}$ whose elements combine by addition and multiplication, subject to the commutative, associative, and distributive laws. The elements also form a group when combined by addition, so that subtraction is always possible in $\mathfrak{A}$, but division, the inverse of multiplication, is not always possible and when possible is not necessarily unique. The unit element of the additive group we denote by 0. We also assume the existence of a *unit* $U$ in $\mathfrak{A}$, *i. e.*, an element such that the equation $UX=0$ in $\mathfrak{A}$ admits only the solution $X=0$. Division by a unit is always possible in $\mathfrak{A}$ and is uniquely determined.† Again $\mathfrak{A}$ contains a sub-algebra of elements simply isomorphic to the set of integers $0, 1, \ldots, m-1$, taken modulo $m$.‡ The elements in $\mathfrak{A}$ corresponding to 0 and 1 are denoted by these symbols. Papers closely related to the subject in hand have been written by Vandiver,† Dickson,§ and Fraenkel.‖ The last of these papers has many results analogous to those of the following pages, but restricts itself to algebras $\mathfrak{A}$ all of whose elements satisfy the unique factorization law.

## I. *The Finite Algebra* $\mathfrak{A}$ *As a System of Residue Classes.*

In a previous paper¶ the writer discussed the group properties of the residue classes belonging to a modular system $\mathfrak{M}=(\mathfrak{m}_n, \ldots, \mathfrak{m}_1, \mathfrak{m})$, where $\mathfrak{m}$ is an ideal of the algebraic field $\Omega$ of degree $k$, while

$$\mathfrak{m}_i=(\psi_1^{(i)}, \ldots, \psi_j^{(i)}, \ldots, \psi_{j_i}^{(i)}),$$

where each $\psi_j^{(i)}$ is a rational integral function of $x_i$ with coefficients that are in turn rational integral functions of $x_1, x_2, \ldots, x_{i-1}$, with coefficients

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* Presented to the American Mathematical Society, December 27, 1915.

† H. S. Vandiver, *Transactions of the American Mathematical Society*, Vol. XIII (1912), pp. 293–304.

‡ L. E. Dickson, "Linear Groups," p. 9.

§ L. E. Dickson, *Transactions of the American Mathematical Society*, Vol. VI (1905), pp. 344–348.

‖ A. Fraenkel, "Ueber die Teiler der Null und die Zerlegung von Ringen," *Crelle's Journal*, Vol. CXLV (1915), pp. 139–175.

¶ Kircher, "Group Properties of the Residue Classes of Certain Kronecker Modular Systems," etc., *Transactions of the American Mathematical Society*, Vol. XVI (1915), pp. 413–434. This reference will be cited as K in the following pages.

that are integers in $\Omega$. We assume the number of residue classes to be finite, which is always true if the following conditions are fulfilled:

(a) Every $\psi_j^{(i)}$ is of finite degree in $x_i$;

(b) Every $\mathfrak{m}_i$ contains at least one $\psi_j^{(i)}$ whose highest power of $x_i$ has a coefficient $\alpha$ relatively prime to the modular system

$(\mathfrak{m}_{i-1}, \mathfrak{m}_{i-2}, \ldots, \mathfrak{m}_1, \mathfrak{m})$, *i. e.*, $(\alpha, \mathfrak{m}_{i-1}, \mathfrak{m}_{i-2}, \ldots, \mathfrak{m}_1, \mathfrak{m}) = (1)$.*

(c) For all values of $i$ we have $j_i > 0$ and finite.

The residue classes are formed by all rational integral functions in $x_1, \ldots, x_n$, with coefficients that are integers in $\Omega$, taken modulo $\mathfrak{M}$. Such a set of residue classes evidently forms a finite algebra of the type under consideration. We shall now show that every finite algebra $\mathfrak{A}$ satisfying the conditions laid down can be represented by the residue classes of a modular system $\mathfrak{M}$. The existence and proof of this correspondence was first brought to the writer's attention by H. S. Vandiver.

We know that $\mathfrak{A}$ contains a sub-algebra of integral marks which we may denote by 0, 1, ...., $m$—1. If there exists another element $x_1$ of $\mathfrak{A}$ not contained in the above set, it follows that $\mathfrak{A}$ contains the marks

$$\alpha_0 x_1^r + \alpha_1 x_1^{r-1} + \ldots + \alpha_r,$$

where the $\alpha$'s range over 0, 1, 2, ...., $m$—1. Since $\mathfrak{A}$ is finite it follows that in the set $x_1, x_1^2, \ldots, x_1^k, \ldots$ there must exist an element $x_1^k$ such that there exists a relation of the type $x_1^k = \beta_1 x_1^{k-1} + \beta_2 x_1^{k-2} + \ldots + \beta_k$, which can be written in the form

$$x_1^k + \gamma_1 x_1^{k-1} + \gamma_2 x_1^{k-2} + \ldots + \gamma_k = 0, \qquad \beta_i = -\gamma_i, \tag{I}$$

where the $\gamma$'s are integral marks. Let $k$ be the smallest integer for which a power of $x_1$ will satisfy a relation of this type. Then all polynomials in $x_1$ are included in the set $\beta_1 x_1^{k-1} + \beta_2 x_1^{k-2} + \ldots + \beta_k$, the $\beta$'s being integral marks. This does not preclude the existence of relations of lower degree between the powers of $x_1$ provided the coefficient of the highest power of $x_1$ in this relation is not prime to $m$. Such a relation is of the form

$$\delta_1 x_1^{k-t} + \delta_2 x_1^{k-t-1} + \ldots + \delta_{k-t} = 0. \tag{II}$$

For brevity denote relation (I) by $\psi_1^{(1)}$, and the various relations (II) by $\psi_2^{(1)}, \ldots, \psi_j^{(1)}, \ldots, \psi_{j_1}^{(1)}$. If there exists in $\mathfrak{A}$ another element $x_2$ that cannot be written as a polynomial in $x_1$ alone, we can show in a similar manner that all elements of $\mathfrak{A}$ that can be obtained from $x_1$, $x_2$, and the integral marks, are of the form $\varepsilon_1 x_2^t + \varepsilon_2 x_2^{t-1} + \ldots + \varepsilon_t$, where the $\varepsilon$'s range over all distinct elements of the form $\beta_1 x_1^{k-1} + \beta_2 x_1^{k-2} + \ldots + \beta_k$. Similarly, there exists a relation

*K, see theorem, p. 425.

$\psi_1^{(2)}=x_2^m+\sigma_1 x_2^{m-1}+\ldots.+\sigma_m=0$, and possibly some relations of lower degree in $x_2$ which we may designate by $\psi_2^{(2)}, \ldots., \psi_{j_2}^{(2)}$. The coefficients of the highest powers of $x_2$ in these last relations are not relatively prime to the modular system $(\psi_1^{(1)}, \ldots., \psi_j^{(1)}, \ldots., \psi_{j_1}^{(1)}, m)$. This process may be repeated until all elements of $\mathfrak{A}$ have been exhausted. By referring to a theorem previously proved * we see that the totality of relations $\psi_j^{(i)}=0$ so obtained together with $m=0$ defines a modular system whose residue classes correspond to the elements of $\mathfrak{A}$. We therefore have the

THEOREM: *The elements of the finite algebra $\mathfrak{A}$ may be represented by the residue classes of a modular system $\mathfrak{M}=(\mathfrak{m}_n, \mathfrak{m}_{n-1}, \ldots., \mathfrak{m}_i, \ldots., \mathfrak{m}_1, \mathfrak{m})$, where $m$ is a rational integer and $\mathfrak{m}_i=(\psi_1^{(i)}, \ldots., \psi_j^{(i)}, \ldots., \psi_{j_i}^{(i)})$, where each $\psi_j^{(i)}$ is a rational integral function of $x_1, x_2, \ldots., x_i$ with coefficients that are rational integers, at least one $\psi_j^{(i)}$ in each $\mathfrak{m}_i$ having a term of form $x_i^{a_i}$ representing the highest power of $x_i$ in that $\psi_j^{(i)}$.*

We are now able to study the properties of the abstract elements of our algebra by studying the concrete residue classes of a modular system representing that algebra. This fact will often prove to be of help in the following pages.

For the necessary definitions and properties of modular systems we refer the reader to the reference *K* previously cited.† There he will find the definition of such terms as: $\mathfrak{M}$ contains the modular system $\mathfrak{M}'$, $\mathfrak{M}$ is divisible by $\mathfrak{M}'$, $\mathfrak{M}$ is an irreducible, an absolute prime, or a simple system. In Fraenkel's paper the term $\mathfrak{M}$ contains $\mathfrak{M}'$ is the same as $\mathfrak{M}$ is divisible by $\mathfrak{M}'$.

Let the elements of $\mathfrak{A}$ be represented by the residue classes of a modular system $\mathfrak{M}$ in the domain of integrity of $(1, x_1 \ldots., x_n)$. If $\delta$ and $\varepsilon$ are any two polynomials in this domain and $C_\delta$ and $C_\epsilon$ are the residue classes containing them, then the operations of addition, subtraction, and multiplication in $\mathfrak{A}$ are evidently defined by the relations:

$$C_\delta+C_\epsilon=C_{\delta+\epsilon}, \qquad C_\delta-C_\epsilon=C_{\delta-\epsilon}, \qquad C_\delta C_\epsilon=C_{\delta\epsilon}.$$

In general, the letters $A, B, C$, etc., denote elements of $\mathfrak{A}$, but units are often denoted by $U$. The significance of 1 and 0 has already been explained. We say that $A$ is *divisible* by $B$ if there exists an element $C$ in $\mathfrak{A}$ such that $A=BC$. From the definition of a unit it follows that the product of a $U$ into each element of $\mathfrak{A}$ gives back all elements of $\mathfrak{A}$, and therefore, by a theorem previously proved,‡ we have the modular system $(U, \mathfrak{M})=(1)$, *i. e.*, a unit of $\mathfrak{A}$ is *relatively prime* to $\mathfrak{M}$. Two elements of $\mathfrak{A}$ that differ from another by a unit factor only are said to be *equivalent* and for purposes of factorization will be regarded as essentially the same. We write $A\equiv 0$, mod $\mathfrak{M}'$, if every

* K, p. 425. † K, pp. 415–418. ‡ K, pp. 431–432.

polynomial $\xi$ in the residue class $A$ of the system $\mathfrak{M}$ satisfies the relation $\xi \equiv 0$, mod $\mathfrak{M}'$. In particular, if $A \equiv 0$, mod $\mathfrak{M}$, we have $A = 0$ in $\mathfrak{A}$. This also defines the meaning of $A \equiv B$, mod $\mathfrak{M}'$. Since $(U, \mathfrak{M}) = (1)$ it follows that $\mathfrak{M}$ contains no system $\mathfrak{M}'$ such that $U \equiv 0$, mod $\mathfrak{M}'$, where $U$ is any unit of $\mathfrak{A}$. When $\mathfrak{M}$ is an absolute prime modular system the algebra $\mathfrak{A}$ reduces to a Galois field as studied by E. H. Moore and Dickson.*

## II. *The Standard Forms of Elements in* $\mathfrak{A}$.

It has previously been proved† that we have:

THEOREM: *If* $\mathfrak{M} = \prod_{i=s}^{i=1} \mathfrak{Q}_i$, *where the* $\mathfrak{Q}_i$ *are the simple factors of* $\mathfrak{M}$, *and if* $f_i$, $i = 1, 2, \ldots, s$, *are any polynomials in* $x_1, x_2, \ldots, x_n$, *with rational integral coefficients, then there exist polynomials* $f$ *such that* $f \equiv f_i$, *mod* $\mathfrak{Q}_i$, $i = 1, 2, \ldots, s$, *and these polynomials are all congruent each to each modulo* $\mathfrak{M}$.

The different polynomials $f$ of this theorem evidently form a residue class of $\mathfrak{M}$ and represent an element $A$ of $\mathfrak{A}$. If $\mathfrak{M} = \mathfrak{Q}_1\mathfrak{Q}_2 \ldots \mathfrak{Q}_i \ldots \mathfrak{Q}_s$ represents $\mathfrak{M}$ factored into its simple modular factors we shall say that $A_i$ is a *simple* element of type $i$ in $\mathfrak{A}$ if every polynomial $\xi$ in the residue class representing $A_i$, mod $\mathfrak{M}$, satisfies the relations $\xi \equiv 1$, mod $\mathfrak{Q}_j$, $j = 1, 2, \ldots, s$, $i \neq j$, while no restriction is made concerning the congruence taken modulo $\mathfrak{Q}_i$. If $A$ is any element of $\mathfrak{A}$ it follows at once from the proof used in the third section of the reference $K$ that there exists one and only one simple element $A_i$ in $\mathfrak{A}$ satisfying the relations

$$A_i \equiv A, \text{ mod } \mathfrak{Q}_i, \quad A_i \equiv 1, \text{ mod } \mathfrak{Q}_j, \qquad j = 1, 2, \ldots, s; \; i \neq j.$$

Hence the product $A_1A_2 \ldots A_s$ is congruent to $A$ modulo every simple factor of $\mathfrak{M}$ and therefore modulo $\mathfrak{M}$. Hence $A = A_1A_2 \ldots A_s$ in $\mathfrak{A}$. From the theory of residue classes we also see that this factorization into simple elements is uniquely determined. Hence we have:

THEOREM: *Every element of a finite algebra* $\mathfrak{A}$ *can be uniquely represented as a product of simple elements.*

We shall represent the factorization of $A$ into simple factors by the notation $A = A_1A_2 \ldots A_s$. If $A$ is a simple element $A_i$ of type $i$ we may write $1_1 1_2 \ldots 1_{i-1} A_i 1_{i+1} \ldots 1_s$, where $1_i = 1$ for all values of $i$. By $0_i$ we understand an element in $\mathfrak{A}$ that is congruent to 0, mod $\mathfrak{Q}_i$, and congruent to 1, mod $\mathfrak{Q}_j$, $j = 1, 2, \ldots, s$, $i \neq j$. Although $1_i = 1$, $0_i \neq 0$. Evidently we have

$$1 = 1_1 1_2 \ldots 1_s \text{ and } 0 = 0_1 0_2 \ldots 0_s.‡$$

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* E. H. Moore, *Bulletin New York Mathematical Society*, Vol. III (1893), p. 75; L. E. Dickson, "Linear Groups," Teubner, 1901.

† K, p. 433.

‡ The results of this section should be compared with Hensel, "Zahlentheorie," pp. 86–92.

We shall define two non-unit elements $A$ and $B$ as *relatively prime* in $\mathfrak{A}$ if there exist two other elements $C$ and $D$ such that $AC+BD=1$. Suppose $A=A_1A_2\ldots.A_i\ldots.A_s$ when written as a product of simple factors. Choose $C=C_1C_2\ldots.C_i\ldots.C_s$ so that $C_i=0_i$ whenever $(A_i, \mathfrak{Q}_i)\neq(1)$, and $A_iC_i=1$ whenever $(A_i, \mathfrak{Q}_i)=(1)$. It has previously been proved that in the latter case the relation $A_iC_i=1$ is always possible.* Evidently we have $A_i0_i=0_i$. Choose $D=D_1D_2\ldots.D_i\ldots.D_s$ so that $D_i=0_i$ whenever $A_iC_i=1$. This includes all cases where $(B_i, \mathfrak{Q}_i)\neq(1)$, for otherwise the residue classes corresponding to both $A$ and $B$ are congruent to 0 modulo the absolute prime modular system contained in $\mathfrak{Q}_i$, and this would preclude any relation of the form $AC+BD=1$ in $\mathfrak{A}$. For all other values of $i$ let $B_iD_i=1$. We see at once that when $C$ and $D$ are so chosen we have $AC+BD\equiv1$, mod $\mathfrak{Q}_i$, $i=1, 2, \ldots., s$, one term always being congruent to 1, the other to 0. Hence, we see from the first theorem of this section that $AC+BD=1$. Hence:

THEOREM: *When two elements $A$ and $B$ of $\mathfrak{A}$ are relatively prime there exist two elements $C$ and $D$ such that $AC+BD=1$, where $C$ always satisfies one of the relations $(C, \mathfrak{Q}_i)=(1)$ or $(C, \mathfrak{Q}_i)=\mathfrak{Q}_i$, $i=1, 2, \ldots., s$, and the same is true of $D$.*

The expression $A_1\ldots.A_i\ldots.A_s$ is the standard multiplicative form of $A$. Fraenkel obtained an analogous result but his proof is no longer valid for our more general case.† We shall now proceed to obtain a standard additive form. By a *component* element of type $i$, written $A_i'$, we understand an element of $\mathfrak{A}$ such that $A_i'\equiv0$, mod $\mathfrak{Q}_j$, $j=1, 2, \ldots., s$, $i\neq j$, with no restriction upon the case of $\mathfrak{Q}_i$. The standard additive form of an element $A$ in $\mathfrak{A}$ is given by the sum of its components, *i. e.*, $A=A_1'+A_2'+\ldots.+A_i'+\ldots.+A_s'$. A component of type $i$ written in factor form gives the expression

$$A_i'=0_10_2\ldots.0_{i-1}A_i0_{i+1}\ldots.0_s.$$

Hence $A\equiv A_i'$, mod $\mathfrak{Q}_i$, $i=1, 2, \ldots., s$, and our standard additive form is seen to be unique. Hence:

THEOREM: *Every element in $\mathfrak{A}$ is uniquely determined by the sum of its components.*

We see that $A_i\equiv A_i'$, mod $\mathfrak{Q}_i$, but $A_i\equiv1$, and $A_i'\equiv0$, mod $\mathfrak{Q}_j$, $j=1, 2, \ldots., s$, $i\neq j$. Again it is evident that the *i-th* component of $A+B$ is $A_i'+B_i'$. In the product $AB$ we have $A_i'B_j'=0$, $i\neq j$, for this product is congruent to zero modulo every $\mathfrak{Q}_i$. Therefore we have:

THEOREM: *The i-th component of the sum or product of two elements of $\mathfrak{A}$ is equal to the sum or product of the i-th components of the two elements.*

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* K, pp. 423–424.

† Fraenkel, *loc. cit.*, p. 166. See also Hensel, "Zahlentheorie," p. 89.

From the preceding theorems we see that if

$$A=A_1A_2\ldots.A_i\ldots.A_s=A_1'+\ldots.+A_i'+\ldots.+A_s',$$

and

$$B=B_1B_2\ldots.B_i\ldots.B_s=B_1'+\ldots.+B_i'+\ldots.+B_s',$$

then

$$AB=A_1B_1\ldots.A_iB_i\ldots.A_sB_s=A_1'B_1'+\ldots.+A_i'B_i'+\ldots.+A_s'B_s'.$$

From these considerations it follows that if we write the elements of $\mathfrak{A}$ as sums of their components we have a finite linear algebra with $s$ units $E_i$, where $E_i=0_1\ldots.0_{i-1}1_i0_{i+1}\ldots.0_s$, such that every element of $\mathfrak{A}$ is of the form $K_1E_1+\ldots.+K_iE_i+\ldots.+K_sE_s$. Here $K_i$ ranges over a set of values simply isomorphic with the set of elements of the algebra $\mathfrak{A}_i$ represented by the residue classes of the modular system $\Omega_i$.* In performing the operations of addition and multiplication we can use the components of type $i$ in place of the elements $K_i$ for all values of $i$, for these two sets of elements are simply isomorphic. Enough has been said to show that $\mathfrak{A}$ is of the type studied by Dickson,† being equal to the sum of $s$ simple algebras having one unit each, the coefficients of these algebras running over different ranges.

## III. *Other Properties of* $\mathfrak{A}$.

We shall not take up the study of the multiplicative groups formed by the elements of $\mathfrak{A}$ since these were treated in the reference cited to which the reader is referred. Turning to the additive groups we see that in many respects they correspond to the modules in the Dedekind theory. In order to obtain a sufficient condition that a set of elements in $\mathfrak{A}$ form an additive group it is merely necessary to take all elements $A$ of $\mathfrak{A}$ that are congruent to zero modulo any system $\mathfrak{M}'$ contained in $\mathfrak{M}$. That this condition is not necessary we see from the group formed by 0 and 1 modulo $(x^2, 2)$. We have, however, the following

THEOREM: *A necessary and sufficient condition that a set of elements in* $\mathfrak{A}$ *form an additive group* $G$ *is that all components of type* $i$, $i=1, 2, \ldots., s$, *found among the elements of the set form a group* $G_i$ *and that there exists no element of* $\mathfrak{A}$ *not in the set whose* $i$*-th component is in* $G_i$ *for all values of* $i$. *The group* $G$ *is equal to the direct product of the groups* $G_1, G_2, \ldots., G_s$.

To prove that the condition is necessary take the group $G'$ formed by all elements of $\mathfrak{A}$. We know all elements of $\mathfrak{A}$ are obtained if in the expression

$$A=A_1'+A_2'+\ldots.+A_i'+\ldots.+A_s'$$

each of the components $A_i'$, $i=1, 2, \ldots., s$, ranges unrestrictedly over all

* Fraenkel, *loc. cit.*, p. 175.

† L. E. Dickson, *Transactions of the American Mathematical Society*, Vol. VI (1905), pp. 344 ff.

values of its set, independent of the others. From this it necessarily follows that $G'$ is the direct product of the $s$ additive groups $H_i$ each of which is composed of all the components of the type designated by the subscript. Since any additive group $G$ of $\mathfrak{A}$ is a subgroup of $G'$ it follows from the theory of groups that $G$ is the direct product of $s$ groups $G_i$, each $G_i$ being that subgroup of its corresponding $H_i$ that is formed by all components of the type indicated by the subscripts that are found in the elements of $G$. It is also clear that there exists no element in $\mathfrak{A}$ but not in $G$ all of whose components are in the various $G_i$, for in forming $G$ as the direct product of the $G_i$ we evidently get an element in $G$ that has exactly this same set of components. This establishes the necessary condition.

To obtain the sufficient condition let us recall that if $A$ and $B$ are two elements of $\mathfrak{A}$ in the set under consideration, and if $A_i'$ and $B_i'$ are their components of type $i$, then $A+B$ has $A_i'+B_i'$ as its component of type $i$. Since $A_i'$ and $B_i'$ are in $G_i$, $i=1, 2, \ldots, s$, the same holds for $A_i'+B_i'$. Therefore it follows from the conditions assumed that $A+B$ must belong to the set. That this set forms a group follows at once for when we add one member of the set to all elements of the set we get back all elements. This proves the theorem.

The multiplication group $G$ formed by the units of $\mathfrak{A}$* furnishes us with an analogue to Dirichlet's general theorem on the units of an algebraic realm, a set of independent generators of the abelian group $G$ corresponding to the linearly independent units of Dirichlet's theorem. To each $U$ corresponds its inverse (or reciprocal) $U^{-1}$ such that $UU^{-1}=1$. Hence the reciprocal of a unit is also a unit. Non-units do not have reciprocals for they are divisors of zero.

We now proceed to define a *prime* element in $\mathfrak{A}$. We say that $A$ is prime in $\mathfrak{A}$ if it cannot be factored into the product of two non-units neither of which is equivalent to $A$. This is the definition found in the theory of algebraic numbers with an added condition. A non-unit that is not a prime can always be factored into two elements neither of which is a unit. Unless both of these factors are primes this process can be repeated. We shall now show that after a finite number of such steps we can always represent such an element of $\mathfrak{A}$ as a product of primes or powers of primes. Suppose the element has been factored into its simple factors which we now proceed to consider separately. It should be remembered that in the following equivalent elements are regarded as essentially the same. Suppose we proceed to factor the simple element $A_i$ into primes, where $A_i$ naturally is a non-unit, for units are not factored. To do this we must first determine the effect of the above-mentioned additional condition given in the definition of a prime. We see that an element $X_i$ in $\mathfrak{A}$ is defined as a prime even when it factors into two non-units, provided at least

* K, pp. 423–424. See also Vandiver, *loc. cit.*, p. 294.

one factor is equivalent to the element. If $U_iX_i$ is the equivalent factor and $X_i=D_iU_iX_i$, write $D_iU_i=B_i$ and proceed to study the equation $X_i=B_iX_i$, where $B_i$ is a non-unit. Throughout we restrict ourselves to simple elements of type $i$. If we take the simple modular factor $\mathfrak{Q}_i$ of $\mathfrak{M}$ associated with our simple elements of type $i$ and write down a descending sequence of modular systems as explained on pages 422 and 423 of the reference $K$ previously cited, we see that the only possible solution for $X_i$ is $0_i$. For, suppose that $X_i\equiv 0$, mod $\mathfrak{Q}_{i,\tau}$, but not modulo $\mathfrak{Q}_{i,\tau-1}$. Since $B_i$ is a non-unit it therefore follows that $X_iB_i=X_i$ is congruent to zero modulo $\mathfrak{Q}_{i,\tau-1}$, which leads to a contradiction except when $X_i=0_i$, where we may take $\mathfrak{Q}_{i,0}=\mathfrak{Q}_{i,1}=\mathfrak{Q}_i$. If $B_i\neq 0_i$ it follows from the reference just cited that there exists a power of $B_i$, $B_i^\epsilon\neq 0_i$, such that $B_iB_i^\epsilon=0_i$. Hence in this case $0_i$ cannot be a prime by definition. This case always occurs except when $\mathfrak{Q}_i$ is an absolute prime modular system, the only case in which there exists no non-unit element of type $i$ outside of $0_i$. Hence in this case we have $0_i=0_i^\alpha$, $\alpha=1, 2, \ldots$, etc., but no other factorization. Therefore our definition of a prime includes

(a) all elements that cannot be factored into two non-unit elements,

(b) the element $0_i$ whenever the corresponding modular system $\mathfrak{Q}_i$ is an absolute prime system.

Let us now proceed to factor $A_i$ and suppose for the present that $A_i\neq 0_i$. If a finite number of steps is not sufficient to factor this element into a product of primes and their powers it follows that since $\mathfrak{A}$ contains but a finite number of simple elements of type $i$, that at least one factor must occur to an arbitrarily high power. Denote this factor by $C_i$. Since $(C_i, \mathfrak{Q}_i)\neq(1)$, there exists by the reference of the preceding paragraph a definite power $\alpha$ of $C_i$ such that $(C_i^\alpha, \mathfrak{Q}_i)=\mathfrak{Q}_i$, which contradicts the fact that $C_i^\alpha$ is a factor of $A_i$ while $A_i\neq 0_i$. Hence a finite number of factorizations is sufficient for this case. If $C_i=0_i$ it follows from the preceding paragraph that we can either write $0_i$ as the product of two non-units neither of them equal to $0_i$, and then proceed as before, or we have $0_i$ equal to a prime by definition. Hence every element of $\mathfrak{A}$ can be written as the product of prime factors. But this factorization is not unique as we see from the example

$$x_2^2\equiv x_2x_2\equiv(x_1+x_2)^2, \mod (x_1^3, x_2^2, 2),$$

where we can easily verify that both $x_2$ and $x_1+x_2$ are primes in the algebra defined by $(x_1^3, x_2^2, 2)$. In this respect our algebra differs essentially from the type studied by Fraenkel. Two primes of $\mathfrak{A}$ need not necessarily be relatively prime, a fact that might be expected from the failure of the unique factorization law. In fact this presupposes the condition $AC+BD=1$ which cannot hold in the above example since $A=x_2$ and $B=x_1+x_2$ are both congruent to 0, mod $(x_2, x_1, 2)$, so that the left-hand side of $AC+BD=1$ is congruent to 0,

mod $(x_2, x_1, 2)$, while this cannot be true of the right-hand side. It furthermore follows that no two primes of an algebra defined by a simple modular system can be relatively prime. Hence, we have the following

THEOREM: *In the finite algebra corresponding to the modular system* $\mathfrak{M}=\mathfrak{Q}_1\mathfrak{Q}_2\ldots\mathfrak{Q}_s$ *there exists s and no more than s primes that are relatively prime each to each.*

Any prime in $\mathfrak{A}$ is equivalent to a prime that is also simple. The first $\alpha$ powers of this latter element are distinct elements of $\mathfrak{A}$, but all later ones are equal to the $\alpha$-th power which is of the form $1_1\ldots1_{i-1}0_i1_{i+1}\ldots1_s$. The first prime when raised to powers has its first $\alpha$ powers essentially different, but all following ones are of the form $U_1\ldots U_{i-1}0_iU_{i+1}\ldots U_s$, where the various $U_j$ run over subgroups of the group of units. Hence, we get:

THEOREM: *The first* $\alpha$ *powers of a simple prime in* $\mathfrak{A}$ *are essentially different, those following are identical with the* $\alpha$*-th power which equals* $0_i$. *If a prime is not simple its first* $\alpha$ *powers are essentially different, while all following powers form a repeating cycle of equivalent elements.*

The algebra $\mathfrak{A}$ also contains sub-algebras. Such can be obtained by taking all elements of $\mathfrak{A}$ that have certain components, say the last $s$–$t$, equal to 0, *i. e.*, in product form these elements can be written $A_1\ldots A_t0_{t+1}0_{t+2}\ldots0_s$. The 0 of $\mathfrak{A}$ is always in such a sub-algebra, but this is never true of any unit of $\mathfrak{A}$. We may, however, let $1_1\ldots1_t0_{t+1}\ldots0_s$ take the place of 1, and let the units of the sub-algebra be those elements whose first $t$ simple factors are units of $\mathfrak{A}$. This sub-algebra is simply isomorphic to the algebra $\mathfrak{A}'$ corresponding to $\mathfrak{M}'=\mathfrak{Q}_1\mathfrak{Q}_2\ldots\mathfrak{Q}_t$.* Comparing this with a necessary and sufficient condition that a set of elements in $\mathfrak{A}$ form a multiplicative group† we have:

THEOREM: *There exists a* (1, 1) *correspondence between* $2^\lambda$ *sub-algebras of* $\mathfrak{A}$ *and the* $2^\lambda$ *multiplicative groups in* $\mathfrak{A}$ *such that the group corresponding to a given sub-algebra is composed of its units.*

In closing we may ask if it is possible to restore the unique factorization law by means of ideals as in the Dedekind theory. This is not the case. Closer study of the subject shows that in such a theory the analogue of an ideal would be the set of all elements of $\mathfrak{A}$ that are congruent to 0 modulo one of the $s$ absolute prime modular systems that $\mathfrak{M}$ contains. In the example already given, namely $\mathfrak{M}=(x_2^2, x_1^2, 2)$, this would give but one (so-called) ideal, and this would be of no avail in solving the factorization of $x_2^2\equiv x_2x_2\equiv(x_1+x_2)^2$, mod $\mathfrak{M}$, uniquely. We can, however, obtain in this manner a number of results analogous to theorems in the theory of algebraic numbers.

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* See Hensel, "Zahlentheorie," pp. 78 ff. † See K, p. 428.

# *The Primitive Groups of Class* 15.

By W. A. Manning.

There are but three primitive groups of class $3p$, $p$ an odd prime, in which occur substitutions of order $p$ and degree $3p$.* All are of class 15. One is of degree 16 and order 80. The other two are the triply transitive group of degree 17 and order 4080, and its subgroup of order 240. Corresponding to every whole number $k$ greater than 3 there exist four simply transitive primitive groups of class $3k$. The alternating group of degree $k+2$ permutes the $\frac{1}{2}(k+1)(k+2)$ binary products $ab$, $ac$, ...., according to a simply transitive primitive group of class $3k$ $(k>3)$.† The three groups ‡

$$(aa'\cdot bb'\cdot \ldots \cdot kk')\,(ab\ldots\ldots j) \text{ pos } (a'b'\ldots\ldots j') \text{ pos},$$
$$(aa'\cdot bb'\cdot \ldots \cdot kk')\,\{(ab\ldots\ldots j) \text{ all } (a'b'\ldots\ldots j') \text{ all}\}\text{pos},$$
$$(ab'ba'\cdot cc'\cdot dd'\cdot \ldots \cdot kk')\,\{(ab\ldots\ldots j) \text{ all } (a'b'\ldots\ldots j') \text{ all}\}\text{pos},$$

are maximal subgroups of

$$(aa'\cdot bb'\cdot \ldots \cdot kk')\,(ab\ldots\ldots k) \text{ pos } (a'b'\ldots\ldots k') \text{ pos},$$
$$(aa'\cdot bb'\cdot \ldots \cdot kk')\,\{(ab\ldots\ldots k) \text{ all } (a'b'\ldots\ldots k') \text{ all}\}\text{pos},$$
$$(ab'ba'\cdot cc'\cdot dd'\cdot \ldots \cdot kk')\,\{(ab\ldots\ldots k) \text{ all } (a'b'\ldots\ldots k') \text{ all}\}\text{pos},$$

respectively. The latter are therefore simply isomorphic to primitive groups of degree $k^2$, and of order $\frac{1}{2}(k!)^2$, $(k!)^2$, $(k!)^2$, respectively. They are simply transitive and of class $3k$. *When $k$ is* 5, *these three and the alternating group of degree* 7 *written on twenty-one letters are the only primitive groups of class* 15 *that contain no substitution of order* 5 *and degree* 15.

Those primitive groups of class 15 which contain a substitution of order 5 and degree 15 are known. There remain to be determined those in which the substitutions of degree 15 are all of the type

$$s_1 = a_1a_2a_3\,.\,b_1b_2b_3\,.\,c_1c_2c_3\,.\,d_1d_2d_3\,.\,e_1e_2e_3\,.$$

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* *Transactions of the American Mathematical Society*, Vol. VI (1905), p. 42.

† C. Jordan, *Comptes Rendus*, Vol. LXXV (1872), p. 1754; American Journal of Mathematics, Vol. XXXII (1910), p. 256, where the error in the formula for the class is corrected.

‡ Cayley, *Quarterly Journal of Mathematics*, Vol. XXV (1891), p. 71, where this notation is explained. For the three groups, see Miller, American Journal of Mathematics, Vol. XXI (1899), p. 299, Theorems III and IV.

Since there exists but one group of order 15, the cyclic group, the only other substitution on the same fifteen letters as $s_1$ is its square. Among all the substitutions in one of our groups $G$ that are similar to $s_1$, let those substitutions be selected which displace a minimum number of letters new to $s_1$, $s_1^2$ excluded. First, it is easy to see that this minimum can not exceed five, the number of cycles in $s_1$. For if such a substitution $s_2$ had more than five letters new to $s_1$, two of them would be found adjacent in one of its cycles. The transform $s_2^2 s_1 s_2$ would then displace a smaller number of letters new to $s_1$ than the assigned minimum, and this is possible only if $s_2^2 s_1 s_2$ displaces no letters new to $s_1$, that is, is $s_1$ or $s_1^2$. But $s_2$, a substitution of odd order, can not transform $s_1$ into its inverse. Then $s_1$ is invariant in the subgroup $H$ generated by the complete set of conjugates under $G$ to which it belongs, and since one of these conjugates is invariant in $H$, all are invariant in $H$ and $H$ is Abelian. But $H$ is not a regular group and hence, being Abelian, is not transitive, and can not be invariant in the primitive group $G$. *Then a substitution $s_2$ (not $s_1^2$) that displaces in its cycles the least possible number of letters new to $s_1$, and is similar to $s_1$, displaces at most five such new letters.*

If $s_2$ has two of these new letters in one cycle, $s_2^2 s_1 s_2$ leaves one of them fixed, and hence is $s_1$. Now since $s_2$ is commutative with $s_1$, it displaces exactly three new letters, and they form one of the five cycles of $s_2$. If $s_2$ does not permute cycles of $s_1$, either $s_1 s_2^2$ or $s_1 s_2$ is of degree less than 15, and obviously not identity. Then $s_2$ permutes cyclically three cycles of $s_1$. If $s_2$ has not this form, it has at most one new letter to a cycle. Let the group $\{s_1, s_2\}$ when not Abelian be called $D$. Of every substitution of $D$ it may be remarked that it either permutes the letters new to $s_1$ only among themselves, or else replaces all of them by letters of $s_1$. Were this not true a transform of $s_1$ could at once be found that would displace at least one new letter, and at the same time fewer new letters than $s_2$. Any substitution of $D$ that replaces one new letter by another new letter transforms $s_1$ into $s_1$ or $s_1^2$. A transitive constituent of $D$ that involves more than one of the letters new to $s_1$ is imprimitive, and one of its systems of imprimitivity is composed of its new letters. If $s_2$ is not similar to $s_1$ in the letters of each of its transitive constituents, $t_1 = s_2^2 s_1 s_2$ is similar to $s_1$ in those letters and may be used for $s_2$ in case $\{s_1, t_1\}$ has the same transitive constituents as $\{s_1, s_2\}$. But if $t_1$ is not similar to $s_1$ in the new (and smaller) transitive sets we may use $t_2 = t_1^2 s_1 t_1$, $t_3 = t_2^2 s_1 t_2$, or a final transform which is similar to $s_1$ in the various constituents of the group generated by it and $s_1$. This apparently fails when $t_1$, $t_2$, or one of these transforms is commutative with $s_1$, but then we may use this Abelian group of

degree 18. *Hence $G$ contains a non-Abelian group $D$ generated by two similar substitutions $s_1$ and $s_2$ of degree* 15, *which are similar in the letters of each constituent, or an Abelian group $\{s_1, s_2\}$ of order* 9 *and degree* 18. In no case do $s_1$ and $s_2$ have more than one common cycle, for as many as two common cycles cause the product $s_1 s_2^2$ to displace fewer than fifteen letters. It is evident that the number of transitive constituents in $D$ does not exceed 5. If there are exactly five constituents, $D$ is either a simple isomorphism between five tetraedral groups or is a (1, 4) isomorphism between a cyclic group of degree and order 3, and an intransitive tetraedral group of degree 16, which is itself a simple isomorphism between four alternating groups of degree 4. Hence, if $s_2$ displaces less than four letters new to $s_1$, it connects cycles of $s_1$.

If $D$ is of degree 16 or 17 it is transitive or is a simple isomorphism between transitive constituents. If $D$ is of degree 17, 18, or 19, it is intransitive. For if $D$ is transitive, the new letters in $s_2$ form a system of imprimitivity of $D$, which is possible only when there are three of them permuted according to a transitive group of degree 6. Since the two generators $s_1$ and $s_2$ are of degree 15, neither can do more than permute cyclically three of the six systems. Then they permute different systems and the group in the systems is intransitive, and $D$ is intransitive.

If $D$ is a transitive group of degree 16, it is of order 48 and has an invariant subgroup of degree and order 16. This group can not be contained in a larger group of degree 16, with the present limitations as to the class of $G$, nor is it contained in any primitive group of degree 17, 18, 19, or 20, if we may rely upon the accuracy of the lists of the primitive groups of these degrees.* If $D$ is a subgroup of a doubly transitive group of degree 21, it must have five systems of imprimitivity of four letters each with a given letter, the new letter $x$, say, in common.† Now $s_1$ permutes among themselves the letters of the systems of imprimitivity to which $x$ belongs, and so far as $s_1$ is concerned $x$ may be a part of as many systems of four letters each as $s_1$ has cycles, but since $s_2$ must replace every system involving $x$ by another system, the two letters of $s_1$ in a cycle of $s_2$ with $x$ can neither of them belong to any system of which $x$ is a part. Hence $D$ can not be five-fold imprimitive. It can not have systems of imprimitivity of two or eight letters because of its two generators. Then the primitive groups under consideration do not contain a transitive subgroup of degree 16 and order 48. If $D$ is intransitive the constituent involving the new letter $x$ is of degree $3k+1$ and order $3(3k+1)$,

* See AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1913), p. 229, for references.

† *Transactions of the American Mathematical Society*, Vol. VII (1906), p. 500.

$k=1, 2, 3, 4$, but a non-regular constituent of degree 7, 10, or 13 is clearly out of the question.* A tetraedral group may be written as a transitive group on twelve and on six letters. But if $D$ is set up as a simple isomorphism between three tetraedral groups of the degrees 4, 6 and 6, respectively, its class is 12.

There remains the simple isomorphism between two tetraedral groups of the degrees 4 and 12. This is not a subgroup of a primitive group of degree less than 21, of the type we are discussing. There exists in $G$ a substitution $s_3$ similar to $s_1$ which directly unites the two transitive constituents of $D$, and has, if it is assumed that no substitution similar to $s_1$ having this property displaces fewer letters new to $D$, in no cycle more than one letter new to D.† Then the group $H=\{D, s_3\}$ is transitive. Its degree does not exceed 21. Every substitution in it replaces a letter of the transitive constituent of degree 12 in $D$ by the same or another letter of that constituent, so that any substitution of $H$ that replaces a letter $y_1, \ldots,$ new to $D$, by a letter $y$, certainly fails to juxtapose letters of the two constituents of $D$. In particular, the subgroup of $H$ that leaves one letter fixed is intransitive. Within this subgroup, which we may call $H_1$, the subgroup $(F)$ generated by all the substitutions of order 3 and degree 15 is invariant. $F$ is positive and all its constituents are positive. It is known that $H$ is not of degree 17 or 19, nor is $H$ an imprimitive group of degree 16, for then its order is of necessity 48, and this case has been examined. If $H$ is of degree 18, and if the two letters $y_1, y_2$ do not constitute a system of imprimitivity, the subgroup $F$ is of degree 17, and has two transitive constituents, which may be of the degrees indicated by the partitions 13, 4 and 12, 5. The presence in $G$ of a substitution of order 13 is impossible. Since by hypothesis $G$ contains no substitution of order 5 and degree 15, the partition 12, 5 is not admissible. But $H$ can not have systems of imprimitivity of two letters each because it is generated by substitutions of degree 15 and order 3. If $H$ is of degree 20, systems of imprimitivity of two or of ten letters each are impossible. The five letters left fixed by $s_1$ can not form a system because $s_2$ displaces one of them. Hence $s_1$ does not permute systems of five letters, and it is clear also that $s_1$ can not permute the letters of each such system among themselves. Hence $H$ is an imprimitive group with five systems of four letters each, which are permuted according to the alternating group of degree 5. Now $D$, because of its constituent of degree 12, certainly permutes systems, in fact can not fix a system to which a letter $y$ belongs unless the four letters $y_1, y_2, y_3, y_4$ are one of the systems of imprimi-

* *Transactions of the American Mathematical Society*, Vol. XVI (1915), p. 139.
† *Transactions of the American Mathematical Society*, Vol. XII (1911), p. 375.

tivity of $H$. That is, there is only one system of imprimitivity in $H$ involving a given letter $y_1$, so that $H$, if it is a subgroup of one of our primitive groups, is to be found in a doubly transitive group of degree 21. Now in $H_1$, the subgroup of $H$ that leaves one letter ($y_1$) fixed, the letters $y_2, y_3, y_4$ can be permuted only among themselves, so that any subgroup of $H_1$ that is similar to $D$ displaces the same letters as $D$ and has the same two sets of letters in its two constituents. If it is assumed distinct from $D$, the group it and $D$ together generate must coincide with $D$ since one constituent of $D$ is already alternating. Then $D$ is unique in $H_1$, and the largest subgroup of $H$ in which $D$ is invariant has a transitive constituent of degree 4.* In the doubly transitive group of degree 21, since $D$ is one of only twenty-one conjugates, the largest subgroup in which $D$ is invariant has a doubly transitive constituent of degree 5. But the group of isomorphisms of the tetraedral group is the symmetric-4 group. Then the required doubly transitive group of degree 21 does not exist. Let $H$ be of degree 21. It can not have systems of imprimitivity of seven letters each since no one of its three generators $s_1, s_2, s_3$ can permute such systems. No letter of $D$ can belong to the same system of three letters with letters $y$. Hence $H$ is primitive. The intransitive subgroup $H_1$, fixing one of the five letters $y$, has no constituent of degree 2,† nor has it a constituent of degree 3.‡ In $H_1$ there is not a constituent of degree 4 composed of letters $y$, for then $D$ would be invariant in $H_1$ and $H_1$ would have a transitive constituent of degree 12. It is obvious that *in any simply transitive primitive group, the degree of one transitive constituent of a subgroup leaving one letter fixed must exceed that of any transitive constituent of a subgroup leaving two or more letters fixed.* It is also true that *the degree of the subgroup that leaves one letter of a simply transitive primitive group fixed is equal to the degree of any one of its subgroups that is generated by a complete set of its similar substitutions.* $F$, for example, is of degree 20. Then the possible degrees of the constituents of $F$ are given by the partitions 15, 5 and 16, 4; 13, 7 and 14, 6 being rejected immediately. If the partition is 15, 5, the second constituent is icosaedral, and $F$ is a (3, 1) or a simple isomorphism between its constituents. If $F$ has an invariant head of order 3, the latter cannot be transformed by substitutions of $H$ into any other subgroup of $F$ because every other subgroup of order 3 in $F$ belongs to a set of at least ten subgroups conjugate under the substitutions of $H_1$. This invariant head of order 3 can not be invariant in

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* *Bulletin of the American Mathematical Society*, 2d series, Vol. XIII (1906), p. 20, Theorem I.

† Miller, *Proceedings of the London Mathematical Society*, Vol. XXVIII (1897), p. 536.

‡ Bennett, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), p. 7.

another subgroup leaving one letter fixed, and therefore is conjugate under substitutions of $H$ to another subgroup of order 3 in $H_1$. But a given subgroup of degree 15 and order 3 is invariant in a subgroup of $H$ that has, in the six letters left fixed by that subgroup, transitive constituents of degree proportional to the number of subgroups in the different sets of conjugates of $H_1$ which in $H$ are united as part of a single set.* If the isomorphism is simple, $D$ is one of five conjugates and by the same theorem should admit an isomorphism of order 5. Then there remains only the partition 16, 4. Obviously $F$ is a (16, 1) isomorphism between its two constituents, the second of which is alternating. $D$ is a subgroup leaving fixed one of the letters of the larger constituent, so that there are four such subgroups, all conjugate in $H_1$. Then there is no primitive group of degree 21 as required.

Let the group $D$ be of degree 17. It is not transitive, nor has it five transitive constituents. Because its degree exceeds its class by only two units, $D$ is a simple isomorphism between its constituents. There is no constituent of degree 3. If there is one constituent of degree 4, $D$ is tetraedral, and there must be one, and hence two, constituents of degree 6, which is absurd when $D$ is of degree 17. There can not be a constituent of degree 7. A constituent of degree 8 must be of such a nature that it can be represented as a transitive group on nine letters. This constituent of degree 9 can not be regular, it can not be of class 7, nor can it be of class 8, for then the two generators of order 3 and degree 9 would be found in a regular self-conjugate subgroup of order 9. But its class certainly exceeds 6. The constituent of lowest degree in $D$ can not be of degree greater than 8, so that finally $D$ is not of degree 17.

If $D$ is assumed to be a transitive group of degree 18 it has six systems of imprimitivity of three letters each. Since $s_1$ (or $s_2$) can not permute more than three of these systems, $D$ is certainly not transitive. It has been shown above that if $s_2$ has two letters new to $s_1$ adjacent in one of its cycles, $D$ is of order 9 with one constituent a transitive group of order 9. It remains to be seen what groups $D$, if any, are present in $G$ in which the three new letters $x$ occur in three different cycles of $s_2$. If $D$ has one constituent of degree 3 (a cycle common to $s_1$ and $s_2$), there is no other constituent of degree 3, and there corresponds to identity of this constituent a subgroup of order 3 and degree 15 in the other letters, so that $D$ is Abelian of order 9, contrary to the assumption that $s_2$ distributes its new letters. Then $D$ is a simple isomorphism between its various transitive constituents. If $D$ has one constituent of degree 4, that is, the alternating group of degree 4, there must be a constituent

**Bulletin of the American Mathematical Society, loc. cit.*

of higher degree, which can only be of degree 6, and then two more constituents of degree 4 each. As a group $D$, this is entirely possible. If it be assumed that the lowest degree of any constituent is 6, there is just one other constituent, of degree 12 and generated by two substitutions of order 3 and degree 9, and whose order is in consequence 36 or a multiple of 36, and not divisible by 5 because the twelve letters fall into four systems of imprimitivity of three letters each. It follows that the constituent of degree 6 is imprimitive. Since it is generated by two substitutions of order 3 and degree 6 it can not have systems of imprimitivity of three letters each. But the imprimitive groups of degree 6 and order 36 or 72 have systems of three letters each.* No constituent can be of degree 7 (10), when $D$ is of degree less than 21 (20). A constituent of degree 8 requires the presence in $D$ of a constituent of degree 10. This closes the search for possible groups $D$ of degree 18, and we now take up the detailed study of the two groups of the orders 9 and 12.

The next step forward will consist in showing that no one of our groups $G$ contains the Abelian group of order 9 and degree 18 as a subgroup. No confusion need arise if we agree to call this group "$D$." There are then in $D$ three transitive constituents of degree 3 and one of degree 9. There exists in $G$, among the substitutions similar to $s_1$, a substitution $s_3$ which connects the largest set of $D$ with some other of the sets of $D$, and which has at most one new letter $y$ in any cycle. Suppose that $H=\{D, s_3\}$ is transitive. If $H$ is of degree 18, it is imprimitive, and has no systems of two, six, or nine letters because of the nature of its generators $s_1, s_2, s_3$. If $H$ has systems of three letters, the letters $x_1, x_2, x_3$ of $s_2$, not in $s_1$, form a system, and any other system involving $x_1$ includes also $x_2$ and $x_3$. But $H$ can not be a subgroup of a doubly transitive group of degree 19. Nor is $H$ of degree 19. Let $H$ be of degree 20. It is imprimitive. Since the two new letters $y_1, y_2$ do not by themselves constitute a system of imprimitivity, we can take from $H$ a substitution $s$ that fixes one of the letters $y_1, y_2$ and displaces the other. The group $\{D, s\}$ separates the letters of the constituents of degree 3 in $D$ from those of the transitive constituent of degree 9. Its order is not divisible by either 5 or 7. Its degree may admit the two partitions, 9, 4, 3, 3 and 9, 4, 6. A constituent of degree 6 is simply isomorphic to the constituent of degree 13. Then there are not two transitive constituents in $\{D, s\}$ of degree 3 each. When the partition is 9, 4, 6, the constituent of degree 4 is alternating, and in (1, 3) isomorphism to the constituent of degree 15. The transitive constituents of degrees 9 and 6 are then each of order 36 and are in simple isomorphism. Now

*Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 287.

in any group of degree 6 and order 36, the subgroup of order 9 is invariant, so that the tetraedral quotient group is absurd. Let $H$ be of degree 21. If $H$ has seven systems of imprimitivity of three letters each, the group in the systems is alternating and involves a substitution of order 5, which in the letters of $H$ is of degree 15. Then $H$ is primitive. The following partitions of the degree of $F$ seem possible: 9, 4, 4, 3, 9, 8, 3, 10, 10. When $H_1$ has a constituent of degree 9, a transform of $F$, because it includes $D$, has a transitive constituent in the same letters. If $F$ has two transitive constituents of degree 10, the first is multiply but not quadruply transitive, and since it is positive, it is one of the groups of class 8, of order 360 or 720. The second constituent, also positive, is not primitive because its class is not greater than 6. But the second constituent can not have systems of two or of five letters. Let $H$ be of degree 22. It is primitive. The possible partitions of the degree of $F$ are, after obvious exclusions, 9, 3, 3, 3, 3, 9, 4, 4, 4, 9, 6, 3, 3, 9, 8, 4, 9, 6, 6, 9, 9, 3, 9, 12, 10, 6, 5, 12, 3, 3, 3, 12, 6, 3, 12, 9. In the first five cases $H_1$ has a transitive constituent of degree 9, and can not be maximal. In the next case $H_1$ has a transitive constituent of degree 3.* Now consider $F$ when its transitive constituents are of the degrees 10, 6, 5. The constituents of degree 10 and of degree 6 are multiply transitive. The constituent of degree 5 is alternating, and the constituent of degree 6, being a simple group, is in simple isomorphism to it. Then $F$ is a simple isomorphism between three icosaedral groups. But all the substitutions of order 3 in such an intransitive group are of degree 18. For it has been noted before in this paper that the icosaedral group when written as a transitive group on ten letters is of class 8, and when on six letters is of class 4. It is now possible to state that 5 does not divide the order of $F$. Any transitive constituent of degree 12 is imprimitive. In the last three partitions, the three letters $y$ must form a system of imprimitivity, for otherwise a certain subgroup $\{D, s\}$ of $F$ would have a transitive constituent of degree 10 or 11. The partition 12, 3, 3, 3 is impossible. Likewise 12, 6, 3 is impossible because the constituents of degree 12 and 6 must be simply isomorphic, while 108 does not divide the order of any group of degree 6. If the constituent of degree 9 (when the partition is 12, 9) is primitive its class is 6, and in consequence its order is 216, a number not divisible by 108. Then the second constituent has systems of imprimitivity of three letters each. These systems are permuted by neither $s_1$ nor $s_2$, nor by any of their conjugates in $F$. Now $F$ has at least one substitution similar to $s_1$ that adds the new system $y_1, y_2, y_3$ to the first constituent, using three or more

*Bennett, *loc. cit.*

cycles for that purpose, and which consequently does not unite the other sets of $D$. Such a group was shown to be impossible in the discussion of the partitions 12, 3, 3, 3 and 12, 6, 3. Then the first constituent of $F$ is of degree 9, and the second of degree 12. Now the subgroup $F'$ of $H_1'$ that fixes a second letter $y$, and which also includes $D$, has a transitive constituent of degree 9. Since the partition 12, 9 has been shown to be impossible, this constituent of degree 9 in $F'$ is on the letters of the largest transitive constituent of $D$. Then $H_1$ is not maximal. Let $H$ be of degree 23.* In this case $H_1$ is transitive and $F$ has two transitive constituents of degree 11. The first being at least triply transitive is of order 7920 and class 8. But the class of the second constituent does not exceed 6.

The group $H'=\{D, s_3\}$ is intransitive. It will be convenient to have before us the actual substitutions of the group $D$. They are:

$$
\begin{aligned}
&1,\\
&s_1s_2=a_1b_2c_3\,.\,a_2b_3c_1\,.\,a_3b_1c_2\,.\,d_1d_3d_2\,.\,e_1e_2e_3\,.\,x_1x_2x_3,\\
&s_1^2s_2^2=a_1c_3b_2\,.\,a_2c_1b_3\,.\,a_3c_2b_1\,.\,d_1d_2d_3\,.\,e_1e_3e_2\,.\,x_1x_3x_2,\\
&s_1=a_1a_2a_3\,.\,b_1b_2b_3\,.\,c_1c_2c_3\,.\,d_1d_2d_3\,.\,e_1e_2e_3,\\
&s_1^2s_2=a_1b_3c_2\,.\,a_2b_1c_3\,.\,a_3b_2c_1\,.\,e_1e_3e_2\,.\,x_1x_2x_3,\\
&s_2^2=a_1c_1b_1\,.\,a_2c_2b_2\,.\,a_3c_3b_3\,.\,d_1d_3d_2\,.\,x_1x_3x_2,\\
&s_1^2=a_1a_3a_2\,.\,b_1b_3b_2\,.\,c_1c_3c_2\,.\,d_1d_3d_2\,.\,e_1e_3e_2,\\
&s_2=a_1b_1c_1\,.\,a_2b_2c_2\,.\,a_3b_3c_3\,.\,d_1d_2d_3\,.\,x_1x_2x_3,\\
&s_1s_2^2=a_1c_2b_3\,.\,a_2c_3b_1\,.\,a_3c_1b_2\,.\,e_1e_2e_3\,.\,x_1x_3x_2.
\end{aligned}
$$

This may be transformed into $\{s_1, a_1b_1c_1\,.\,a_2b_2c_2\,.\,a_3b_3c_3\,.\,d_1d_3d_2\,.\,x_1x_2x_3\}$ by means of $b_1c_1\,.\,b_2c_2\,.\,b_3c_3\,.\,x_2x_3$. The group $D$ is also invariant under all the substitutions of the group

$$\{d_1e_1\,.\,d_2e_2\,.\,d_3e_3\,.\,b_1c_2\,.\,b_2c_3\,.\,b_3c_1\,.\,x_2x_3,\quad d_1x_1\,.\,d_2x_2\,.\,d_3x_3\,.\,a_2b_3\,.\,b_2c_3\,.\,c_2a_3\,.\,e_2e_3,\quad d_1d_2d_3\}.$$

Hence we are at liberty to write $s_3=(a_1d_1—)\,.\,.\,.\,.$ One of the substitutions of degree 15 in $s_3^2Ds_3$ has the property of connecting the set $a_1, \ldots,$ of $D$ and another set. Hence for $s_3$ we may choose a substitution similar in its transitive constituents to one of the substitutions of $D$. Whenever the transitive constituent $a_1, a_2, \ldots,$ of $H'$ is of degree less than 18, every substitution of $H'$ replaces some one of the nine letters $a_1, a_2, \ldots,$ of $D$ by one of those letters, so that it is evident that a substitution of $H'$ which replaces a $y$ by a $y$ (new letters to $D$) does not have a letter of the set $a_1, a_2, \ldots,$ of $D$ in any cycle with one of the remaining letters $d_1, \ldots,$ of $D$. Partitions of the degree of $H'$ not entailing substitutions of order 5 and degree 15, of order 7 and

* Burnside, "Theory of Groups," 2d edition (1911), p. 341.

degree 14, and so forth, are 12, 3, 3, 12, 6, 12, 4, 3, 16, 3, 12, 4, 4, 12, 8, 16, 4, 14, 7, 15, 6, 18, 3, 16, 3, 3, 14, 8, 15, 7, 16, 6, 18, 4, 16, 4, 3, 15, 8, 16, 7, 20, 3. Consider the last case: 20, 3. The constituent of degree 20 is not primitive, nor has it systems of imprimitivity of degree 2 or 10. Only $s_3$ permutes systems to which the letters $y$ belong. Then there are not systems of four letters each. If $s_3$ does not already fix $x_1$, $x_2$, $x_3$, transform $D$ and $s_3$ by the substitution $e_1x_1 \cdot e_2x_3 \cdot e_3x_2 \cdot b_1a_3 \cdot b_2c_3 \cdot a_2c_1 \cdot d_2d_3$ under which $D$ is invariant. Then $s_3$ fixes $x_1$, $x_2$, $x_3$. Then $s_2$ does not permute systems, nor consequently, does $s_1$ if we look at $d_1d_2d_3$, but it does if we look at $a_1b_1c_1$. The partition 20, 3 is impossible. If the partition is 18, 4 or 18, 3 the larger constituent is imprimitive and can only have systems of three letters, which in turn are permuted according to a primitive group of class 3. Here we encounter substitutions of order 5 and degree 15. A transitive constituent of degree 16 in $H'$ involves one or four new letters $y$. Its order is at least 144. Since $G$ can not contain a subgroup of degree 16 of order greater than 16, the partitions 16, 3 and 16, 4 are impossible. Let the partition be 16, 3, 3. $H'$ is a (16, 1) isomorphism between a transitive group of order 144 and a direct product of order 9. The first constituent is not primitive. It has no systems of two or eight letters, hence its four systems of imprimitivity are permuted according to the alternating group of degree 4. The intransitive head of order 12 is also of order 12 in each transitive constituent, hence is a simple isomorphism between four tetraedral constituents. Now the transitive constituents of degree 3 are in the letters $e_1e_2e_3$, and $x_1x_2x_3$. The substitution $s_1^2s_2$ is in the intransitive head of the constituent of degree 16. It is of degree 9 in those letters whereas it should be of degree 12. Let $H'$ have constituents on sixteen and on six letters. Since the only group of degree 6 whose order is divisible by 144 is the symmetric group, $H'$ is not a simple isomorphism. The constituent of degree 6 is alternating because it is generated by three cycles of degree 3. The constituent of degree 16 is primitive and therefore is of order 5760. There are two such groups, but they are both of class 12 while our constituent in $H'$, because of $s_1^2s_2$, is of class less than 12. If the partition is 16, 4, 3, and if we are not to have a subgroup of degree 16 with the factor 3 in its order, the intransitive constituent of degree 7 is a direct product of order 36. The large constituent is of order 144, 288, or 576, as the isomorphism is (4, 1), (8, 1) or (16, 1). This alone shows that the first constituent is imprimitive when of order 144 or 288. Suppose it is primitive of order 576. Its class should be 12, but $s_1^2s_2$ is of degree 9 in the letters of this constituent. This first constituent because of $s_1^2s_2$ can not have systems of imprimitivity of two letters each. Systems of four

letters each are left unchanged by $s_1^2 s_2$ of degree 9 in the leading constituent, and these four systems are permuted according to the tetraedral group. The intransitive head consisting of all the substitutions of the constituent in question that do not permute systems is then of order 12, 24, or 48. A transitive constituent of this head is of degree 4, and is alternating or symmetric. If the order of the head is 12 or 24 it is a simple isomorphism, and all its substitutions of order 3 are of degree 12, a condition violated by $s_1^2 s_2$. If the order is 48 the symmetric group is not involved. There are only two ways by which we can get the order 48 from four alternating groups of degree 4. In one way an intransitive constituent of degree 12 is formed as a simple isomorphism, and that is put in a (4, 4) isomorphism to a fourth alternating constituent. Another way is to put in (4,4) isomorphism two similar intransitive tetraedral groups on eight letters. But in any event the substitutions of order 3 in the head are of degree 12. The constituent of degree 7 is on the letters $e_1 e_2 e_3 x_1 x_2 x_3 y_5$, and is an alternating group. The alternating group of degree 7 can not be written as a transitive group of degree 16. Then it should be a (16, 1) isomorphism between a primitive group of degree 16 and order 40320 and the alternating group of order 2520. But the former group is of class 12 and does not contain its part of $s_1^2 s_2$. Let $H'$ have a transitive constituent of fifteen letters. Its order is divisible by 135. No non-alternating primitive group of degree 15 has the factor 27 in its order. Then a constituent of degree 15 has five systems of imprimitivity of three letters each, permuted according to an icosaedral group. The smaller transitive constituent, containing a substitution of order 5, is an alternating group or the icosaedral group written on six letters. None of the orders of these groups are divisible by 27, and the last not even by 9. A (3,1) isomorphism between the imprimitive group of degree 15 with its icosaedral quotient group and one of these three alternating groups is impossible. Let $H'$ have a leading constituent of degree 14. It is a simple isomorphism. The constituent of degree 14 is not primitive. It must have seven systems of two letters each. But $s_1^2 s_2$ can not respect such systems if $s_3 = a_1 d_1 - \ldots$. The partition 12, 3, 3 demands an absurdity, a simple isomorphism between an intransitive constituent of degree 6 and order 9, and a transitive constituent of degree 12. Let 12, 6 be the next partition for study. The second constituent is imprimitive because 5 can not divide the order of $H'$ in this case. But the group generated by $(e_1 e_2 e_3)$, $(x_1 x_2 x_3)$ and $(e_1 x_1 -) \ldots$, the last of degree 3 or 6 in these letters, is necessarily alternating. Let the partition of the degree of $H'$ be 12, 4, 3. The constituent of degree 7 can not be of order 12, for then the leading constituent is the regular tetraedral group

and all the substitutions of order 3 in $H'$ are of degree 18. Let the intransitive constituent be of order 36, a direct product. It has an invariant subgroup of order 12 and degree 16 which is a simple isomorphism. This subgroup has been fully considered. Let the partition be 12, 4, 4. The group $H'$ is not a simple isomorphism, nor does it involve the direct product of two tetraedral groups. The intransitive constituent of degree 8 is therefore of order 48, and this is the order of $H'$, a number not divisible by 9. The remaining partition is 12, 8. The order of the second constituent of this simple isomorphism is 72 or a multiple of 72. Since it is of class 3, it is not primitive: $\{e_1e_2e_3, x_1x_2x_3\}$ does not respect systems of two letters, and $s_3$ can not permute systems of four letters.

Let $D$ be the group of degree 18 and order 12 for present consideration. There exists a substitution $s_3$ similar to $s_1$ that replaces a letter of the transitive constituent of degree 6 by a letter of another constituent and that has no two new letters, new to $D$, in one cycle. Let $H$ be transitive if possible. If $H$ is of degree less than 21, it is imprimitive. If $H$ is of degree 18, there is no system of imprimitivity of degree 9, 6, or 2. In case there are systems of three letters, the four letters of a constituent of degree 4 in $D$ belong to different systems, while the constituent of degree 6 has systems of two but not of three letters, so that $D$ can not permute systems of three letters when $H$ is of degree 18. Nor could it do so if $H$ were an imprimitive group of degree 21. If $H$ is imprimitive of degree 20, no systems are of two or ten letters. Systems of five letters are also readily seen to be impossible. For if $s_1$ permutes systems, $y_1, y_2, x_1, x_2, x_3$, constitute a system, not respected by $s_2$. Similarly $s_2$ can not permute systems. Finally $H$ can not have systems of four letters each, for the reason that no letter of $D$ can belong to a system of four letters with $y_1$. If $H$ is imprimitive of degree 21, it was noticed above that there can not be systems of three letters, and systems of seven letters are obviously impossible. If $H$ is primitive of degree 21, we can find a substitution $s$, similar to $s_1$, in $H$ which fixes one of the letters $y_1, y_2, y_3$, and displaces at least one of them. This substitution $s$ can not, by hypothesis, connect the constituent of degree 6 of $D$ with another constituent of $D$. The order of $\{D, s\}$ is not divisible by 5 (the degrees of the transitive constituents of $\{D, s\}$ are not all divisible by 5), 7, or 13. Then if $\{D, s\}$ is of degree 19, the transitive constituents in it are of the degrees indicated by the partition 6, 9, 4, which is an impossibility because the constituent of degree 4 is of order 12, and the intransitive constituent of degree 15 can not be in multiple isomorphism to it. If $\{D, s\}$ is of degree 20, $s$ adds two letters to the constituent of degree 6 of $D$.

The degrees of the constituents can not be given by 8, 4, 4, 4, because the order of the intransitive constituents of degree 8 is 12, 48, or 144, while that of the other intransitive constituent in simple isomorphism to it is a multiple of 32. If the partition is 8, 8, 4, $\{D, s\}$ is not a simple isomorphism between its three constituents, since one is alternating of degree 4, but the two constituents of degree 8 are in simple isomorphism to each other. The constituent of degree 8 that involves two new letters permutes these two new letters as a system of imprimitivity, one of four, which go according to the alternating group of degree 4, because the constituent in $D$ admits no other systems than those of two letters, and permutes them in a cycle of order 3. The head involves substitutions of degree 4 (as in $D$), and hence the class of $G$ is lowered. If the degrees of the transitive constituents of $\{D, s\}$ are 8, 12, the above remarks still apply to the first constituent, though the class of $G$ is not necessarily lowered by the presence of substitutions of degree 4 in the head of the constituent of degree 8. The other constituent, in simple isomorphism to the one of degree 8, has the factor 9 in its order because of the substitutions of order 3 and degree 9 in it. This, the smaller constituent, whose order is twelve times a power of 2, does not permit. If $H$, transitive, is of degree 22, it is primitive, and we may again study the group $\{D, s\}$, confining ourselves to the degree 21, since the preceding remarks in regard to the degrees 19 and 20 apply to this new $\{D, s\}$ without change. To be certain that $s$ may be found without three of the new letters $y$ in one cycle, recall that $H$, being primitive, has in it a substitution $t$ which replaces a $y$ by a $y$, and replaces a letter of $D$ by a $y$. Then $t^{-1}Dt$ has no constituent of degree less than 4, fixes one (and only one) $y$, and gives us the substitution $s$ similar to $s_1$ with the three letters $y$ in three cycles. The subgroup $F$, invariant in $H_1$, keeps the letters of the constituent of degree 6 of $D$ and the other letters of $D$ in separate constituents. $F$ is of degree 21. If $F$ has a constituent of degree 9, made by adding three letters $y$ to the constituent of degree 6 in $D$, that constituent is primitive, and $s$ may be chosen so that the degree of $\{D, s\}$ is 19 or 20. The possible partitions of the degree of $F$, which do not obviously lead to substitutions of order 11, of order 7, of order 5 and degree 15, are 6, 5, 5, 5, 6, 10, 5, 6, 9, 6, 6, 15, 8, 9, 4. We note that in $F$, and hence also in the other three subgroups that are conjugate to $F$ and leave one of the four new letters $y$ fixed, a transitive constituent that contains the six letters of the largest transitive constituent of $D$ is of degree 6 or 8, thus compelling the presence of a transitive constituent of degree 6 on the same letters in two subgroups, $F$ and one of its transforms. This is impossible when $H_1$ has a transitive constituent in these six letters, as must be

the case if the partitions are 6, 5, 5, 5, 6, 10, 5, or 6, 15. If the partition is 6, 9, 6, the third constituent is alternating, and $G$ contains a substitution of degree 15 and order 5. If one constituent of $F$ is of degree 4, its order is 12, and the group is not a simple isomorphism between its three transitive constituents, but the constituents of degree 8 and degree 9 are simply isomorphic. The former is imprimitive and its order is not divisible by 9. We conclude that $H$ is not a transitive group of degree 22. If $H$ is of degree 23 it is doubly transitive. The subgroup $F$ of $H_1$ must have its transitive constituents of equal degree, but the degree of one of them can range only from 6 to 10. Hence $H$ is not of degree 23.

The group $H'=\{D, s_3\}$ is intransitive. Since $s_3$ replaces one of the six letters of the largest transitive constituent of $D$ by a letter of that constituent, the constituent of degree 6 in $s_3^2 D s_3$ or $s_3 D s_3^2$ has letters of the constituent of degree 6 and of one of the constituents of degree 4 of $D$ in a cycle of one of its generators of order 3. Then it may be assumed that $H'$ has the three generators of any one of its constituents similar substitutions. The possible partitions of the degree of $H'$ are 10, 5, 5, 10, 10, 12, 4, 4, 12, 8, 15, 5, 12, 9, 16, 4, 16, 5, 12, 5, 5, 12, 10, 18, 4, 18, 5. A transitive constituent of degree 12 is not doubly transitive because there is not another constituent of so high a degree as 11. Then it is an imprimitive group generated by three similar substitutions of order 3 and degree 9, whence it follows that it does not admit systems of two, four or six letters. Its systems of three letters go according to the tetraedral group. Then the order of this constituent is divisible by 144, but not by 5, a remark which disposes of the partitions 12, 5, 5 and 12, 10, and which requires that the intransitive constituent on eight letters in the case 12, 4, 4 be a direct product. But the direct product of two alternating groups of degree 4 can not be written as a transitive group of degree 12. When the partition is 12, 8, the second constituent, generated by two substitutions of degree 6 and order 3, has four systems of imprimitivity of two letters each so that its order is not divisible by 9. But in this case the two constituents are in simple isomorphism. A constituent of degree 9 generated by similar substitutions of degree 6 and order 3 is primitive, and being positive and not triply transitive (to avoid a substitution of order 7), is the group of order 216. Again the two constituents are in simple isomorphism. An absurdity has been reached: 144 does not divide 216. Consider the partitions 18, 4 and 18, 5. The larger constituent is not primitive, and since it is generated by four cycles of $s_1$, $s_2$, and $s_3$, it has nine systems of imprimitivity permuted according to a primitive group. The substitutions of order 2 in $D$ are of degree 4 in these

systems, for they do not interchange systems of two letters in the constituent of $D$ of degree 6, and in the constituents of $D$ of degree 4 no two letters of the same constituent can belong to the same system of imprimitivity. Now the group in the nine systems is not alternating, and there is no primitive group of class 4 of degree greater than 8. If the partition is 15, 5, the larger constituent, with the generators $s_1, s_2, s_3$, can not be imprimitive. If it is primitive it is simply transitive (to avoid a substitution of order 7) and of class greater than 9. There is but one group satisfying these conditions, and it is of order 360, simply isomorphic to the alternating group of degree 6, a simple group. In the case of a transitive constituent of degree 16, systems of imprimitivity of two letters are possible so far as $D$ is concerned. But the two new letters in $s_3$ can not form part of a larger system of four or eight letters. The systems of two letters are therefore permuted according to a primitive group of degree 8, and again we encounter a substitution of order 7. Let $H'$ have two transitive constituents of degree 10. The second constituent of degree 10 is generated by three substitutions of order 3 and degree 6. If the two new letters of $s_3$ do not form a system of imprimitivity, $H'$ has a subgroup $\{D, s\}$ ($s$ is similar to $s_1$ and is determined in the usual way) of degree 19. Among the small number of partitions of which the subgroup $\{D, s\}$ admits only one claims attention: 6, 9, 4. Here the intransitive constituent of $\{D, s\}$ of degree 15 must be in (3, 1) isomorphism to the alternating constituent of degree 4. But the constituent of degree 6 can not have an invariant subgroup of order 3, because of the substitutions of $D$ in those six letters. Then the second constituent of $H'$ has five systems of imprimitivity and its order is sixty times a power of 2. The other constituent is a positive primitive group (see its generators $s_1, \ldots.$) of order 360 or 720, multiples of 9. The primitive group of degree 10 and order 60 was passed over because it is not simply isomorphic to an imprimitive group of degree 10. But one partition is left: 10, 5, 5. In this case the intransitive constituent of degree 10 can only be a simple isomorphism between two icosaedral groups, because the direct product of two icosaedral groups can not be represented as a transitive group on ten letters. Hence the transitive constituent of degree 10 is also of order 60, a simple isomorphism between three icosaedral groups of the degrees 10, 5, and 5.

Because $H'$ contains a substitution of order 5 and degree 20, the highest degree of any primitive group of which it is a subgroup is 25. Let now the substitution $s_4$ be chosen subject to the same conditions as was $s_3$. Let it be further assumed that $H=\{H', s_4\}$ is transitive. $H$ is not a primitive group of degree 20. Since the transitive constituent of degree 10 in $H'$ is primitive,

systems of imprimitivity can only be of two or ten letters each. But such systems can not be permuted by any of the generating substitutions $s_1, s_2, s_3, s_4$. Let $H$ be of degree 21. These generators show that it is not imprimitive. The subgroup $F$, generated by all the substitutions $s_1, \ldots,$ in $H_1$, the subgroup of $H$ that leaves one letter fixed, is by hypothesis intransitive. Now $F$ can not have a constituent of degree 15 unless it includes a substitution similar to $s_1$ that connects the transitive constituent of degree 10 of $H'$ with another constituent of $H'$ and displaces fewer new letters than $s_4$. Suppose that $F$ has two constituents, both of degree 10. The icosaedral group of degree 10 (of class 8) is not contained in a larger group of the same degree and class,* nor in a positive group of degree 10 and class 6. Then $F$ coincides with $H'$. The largest primitive group of degree 10 in which this icosaedral group on ten letters is invariant is of order 120, isomorphic to the symmetric group on five letters. Then the order of $H$ is 1260 or 2520. But consider for a moment the possible order of the largest subgroup of $H$ in which a subgroup of order 7 (and degree 21) is invariant. This subgroup can contain no substitution of order 3 or 2 capable of transforming a substitution of order 7 into a power without permutation of cycles, for all the subgroups of order 3 in $H_1$ are of degree 15, not 18, and subgroups of order 2 are of degree 16. This is evident when $H_1$ is of order 60, but perhaps requires explanation if the order of $H_1$ is 120. In the latter case one constituent of $H_1$ is, as we have seen, the primitive group of order 120 of class 6; it has fifteen substitutions of order 2 and degree 8, and ten substitutions of order 2 and degree 6, the latter corresponding to the ten transpositions of five letters. The other constituent of $H_1$ is then not intransitive (the class must be maintained) but is the imprimitive representation of the symmetric-5 on ten letters. This imprimitive constituent has two systems of five letters each, and also five systems of two letters each. The subgroup that leaves one letter fixed, leaves two fixed, and is the alternating group in the four systems. Then the imprimitive constituent has fifteen substitutions of order 2 and degree 8, and ten substitutions of order 2 and degree 10. In $H_1$, therefore, all the substitutions of order 2 are of degree 16. No substitution of $H$ can transpose two cycles of a substitution of order 7 and leave one cycle unchanged, the class of $H$ being 15. Hence the order of the largest subgroup of $H$ in which a substitution of order 7 is invariant is 7 or 21. Of the four quotients 1260/7, 2520/7, 1260/21, 2520/21, only 120 is congruent to unity, modulo 7. Then $H_1$ is of order 120, and there are in $H$ 120 conjugate subgroups of order 7, each of which is invariant in a group of order 21. In $H$ there are one hundred and twenty-six conjugate subgroups of order 5. An

*AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII (1910), p. 254.

invariant subgroup ($K$) of $H$ is transitive, hence 7 is a divisor of its order, which can only be 840. But it is now clear that $K$ must have in it the one hundred and twenty-six subgroups of order 5, giving to $K$ more than eight hundred and forty operators. Then $H$ is a simple group. It is well known that there is only one simple group of order 2520, the alternating group of degree 7. It has a transitive representation of class 15 on twenty-one letters, most readily set up as the group of the permutations of the twenty-one binary products $ab, ac, \ldots$, of seven letters. It is not contained in a larger primitive group of the same degree and of class 15. Since all its substitutions of degree 15 are conjugate it is not maximal in a transitive group of degree 22.* Let $H$ be of degree 22. Its generators show that it is primitive. The subgroup $F$ is of degree 21, and has the same primitive constituent, the alternating-5 group on ten letters, as $H'$. Since $F$ is a simple isomorphism this is impossible. Let $H$ be of degree 23. It is doubly transitive, and $F$, invariant in the transitive subgroup $H_1$, has two transitive constituents of degree 11 which are interchanged by half the substitutions of $H_1$. The first constituent, in which the primitive group of degree 10 and order 60 is maximal, is quite possible, but we get into difficulties with the second constituent, simply isomorphic to it, which is of class 6 or less because of the subgroup $D$. Next, let $H$ be of degree 24. Because of the primitive constituent of degree 10 in $H'$, and the generators $s_1, \ldots$, of $H$, $H$ is not imprimitive. The only divisions of the letters of $F$ into transitive sets that do not introduce substitutions of order 5 and degree less than 20, of order 7 and degree less than 21, of order 11 and degree less than 22, and cycles of thirteen letters are 11, 12, 12, 11, 12, 6, 5. The 12, 11 partition is impossible because of the substitutions of degree 6 in the second constituent. The non-alternating group of degree 11 when represented on twelve letters is multiply transitive and can not have substitutions of degree 6 in it, as the partition 11, 12 requires. In the case of the partition 12, 6, 5 the first constituent being primitive involves a substitution of order 11. Then $H$ is not of degree 24. Finally the primitive group $G$ that includes $H$ is of degree 25. The order of $G$ is not divisible by 7 or 11. Then all tentative partitions of the degree of $F$ are seen at a glance to be impossible.

The group $H''=\{H', s_4\}$ may be intransitive. If $s_4$ is not similar to $s_1$ in the letters of each of the constituents of $H''$ (as $s_3$ was similar to $s_1$) another substitution from $s_4^2 H' s_4$ will serve for $s_4$ and will be similar to $s_1$ in the two sets of letters. For all subgroups of order 3 are conjugate in $H'$, and *a fortiori* similar in the various sets of letters. Now the degree of the constituent of $H''$ that involves the primitive group of degree 10 and order 60 is at least 15, and

**Bulletin of the American Mathematical Society, loc. cit.*

at most 19. It is therefore primitive and of class 12. Since the second constituent is alternating of degree 5 or 6, we have to seek those positive primitive groups of class 12 and of degree 15 and 16 that can be isomorphic to one of these two alternating groups. If the isomorphism is simple, the larger constituent is the simply transitive primitive group of degree 15 and order 360. If the isomorphism is not simple it is a (16, 1) isomorphism between a primitive group of degree 16 and order 960 or 5760, and one of the given alternating groups. If the constituent of degree 16 is multiply transitive, every substitution of its subgroup that leaves one letter fixed replaces a letter of the transitive constituent of degree 10 of $H'$ by a letter of that constituent, and one substitution at least replaces a letter of that constituent by a letter of a second constituent of $H'$. The transform of $H'$ by this substitution contains a substitution similar to $s_1$ that has the properties of $s_4$ with fewer new letters. Then the constituent of degree 16 can only be the simply transitive group of degree 16 and order 960 (it is $G_7$ in Miller's list of the primitive groups of degree 16*). Then $H''$ is of degree 21 only. All the subgroups of degree 15 in $H''$ are conjugate. Now a substitution $s_5$ is chosen (after the manner of the choosing of $s_4$) which with $H''$ generates a transitive group $H$. Because of the fact that the degree of this group can not exceed 25, and because of the presence of the primitive constituent of degree greater than 14 in $H''$, $H$ is primitive. If its degree is 21, it may be handled by means of the arguments used when $s_1, s_2, s_3, s_4$ was assumed to be a primitive group of degree 21, $H'$ being in $F$ and thus determining the simple group of order 2520, if any. If $H$ is of degree 22, $F$ coincides with $H''$. In $H''$, that is, in $H_1$, all the subgroups of degree 15 form just one set of conjugates, while one of them leaves fixed seven letters which therefore form a transitive set in the largest subgroup of $H$ in which that subgroup is invariant, which latter group, however, can contain no substitution of order 7. Let $H$ be of degree 23, a doubly transitive group. Then $F$ should have two transitive constituents of degree 11. Let $H$ be of degree 24. $F$ has two distinct sets of letters. A constituent of degree 15, isomorphic to the alternating group of degree 8, is doubly transitive and a constituent of degree 16, isomorphic to the alternating group of degree 7, is triply transitive. To show the non-existence of the simply transitive primitive group $H$, we may call attention to the following general theorem, proof of which is appended:

*If the degree of a transitive constituent of the subgroup leaving one letter fixed in a simply transitive primitive group exceeds by two (or more) units the*

*Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XX (1898), p. 231.

*degree of any other transitive constituent of that subgroup, then the constituent of highest degree is a simply transitive group.**

It may be proved in few words. Let $G_1$ fix the letter $x$ of the simply transitive primitive group $G$ of order $g$ and degree $n$, and let $a_1, a_2, \ldots, a_m$ be the letters of the transitive constituent of highest degree in $G_1$, supposed, if possible, to be multiply transitive.

The order of the subgroup of $G_1$ that fixes $a_1$ and $a_2$ is $g/nm(m-1)$, while the order of any subgroup of $G_1$ that fixes two letters, one or both of which do not belong to the set $a_1, a_2, \ldots, a_m$ is greater than $g/nm(m-1)$. Now consider the subgroup $G'$ of $G$ that fixes $a_1$. Since the order of the subgroup of $G$ that fixes $a_1$, $x$, and $a_i$, $i=2, 3, \ldots, m$, is exactly $g/nm(m-1)$, $x$ and $a_i$ must be in transitive constituents of the subgroup that fixes $a_1$ of degree greater than $m-2$; but there is in the subgroup that fixes $a_1$ only one such constituent. Then either $G_1$ is not a maximal subgroup of $G$ or the transitive constituent of highest degree in $G_1$ is but simply transitive. The theorem is proved.

Let $H$ displace twenty-five letters. Its order is not divisible by 7. Since the larger constituent of $F$ (and therefore of $H_1$) is simply transitive, the other constituent is an alternating group and necessarily introduces the factor 7 into the order of $H$.

From now on we shall be concerned only with primitive groups $G$ of class 15 in which no two substitutions of degree 15 have so many as twelve letters in common.

The group $D$ of degree 19 is not transitive. It has no transitive constituent of degree 7, 11, or 15. Then it has one constituent of degree 3, represented by a cycle common to $s_1$ and $s_2$. If $D$ has a constituent of degree 4, it is tetraedral, and $D$ is a simple isomorphism between two intransitive constituents of degree 12 and degree 7. It is of order 12 and has five transitive constituents. If $D$ has a transitive constituent of degree 8, a second transitive constituent of degree 8 is in simple isomorphism to the first. Two substitutions of degree 6 and order 3 that generate an imprimitive group of degree 8, such as we evidently have here, limit the group to systems of imprimitivity of two letters. It can not have systems of four letters. The group in the systems is tetraedral. The head is not the identity and is not of higher order than 2. There is one and only one group satisfying these conditions, $\{135 . 246, 147 . 238\}$ with the head $(12 . 34 . 56 . 78)$.†

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* Cf. Jordan, *Bulletin de la société mathématique de France*, Vol. I (1873), p. 198.

† Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), see pp. 323 and 332.

Let us study this group $D$ of order 24. It has no subgroup of order 12. There is of course in $G$ a substitution $s_3$ similar to $s_1$ that unites two constituents of $D$ and has at most one new letter to a cycle. One of the transforms by $s_3$ of the eight substitutions of order 3 in $D$ certainly unites two transitive constituents of $D$ and may be used for $s_3$. We are thus assured of the advantage of having $s_3$ similar to $s_1$ and $s_2$ in the constituents of $\{D, s_3\}$ whenever it is not transitive.

Let $H=\{D, s_3\}$ be transitive. It is not of degree 19. If $s_3$ displaces just one letter new to $D$, it displaces three or four of the letters of $D$ that are fixed by $s_1$. $H$ is imprimitive and no systems are possible except of four letters, and one such system consists of the four letters of $s_2$ that are new to $s_1$. Three of these letters do not form a cycle of $s_3$, for then the existence of a substitution similar to $s_1$ with not more than two letters new to $s_1$ could be deduced. Since $s_3$ bears this same relation to any one of the four subgroups of order 3 in $D$, it would need to be of degree 16 at least. Let $H$ be of degree 21. It is primitive. In $F$ the letters are partitioned thus: 9, 8, 3, 8, 8, 4. The first places an impossible constituent of degree 3 in $H_1$, and in the second case $H_1$ is not maximal because $D$ is in two subgroups $F$ and $F'$, and thus determines the letters of one or two constituents of $H_1$ and $H_1'$. If $H$ is of degree 22 it is primitive. The partitions of the degree of $F$ are 9, 9, 3 and 9, 8, 4. Here there is no constituent of degree 3 in $H_1$. The constituent of degree 8 is of order 24 and should be simply isomorphic to the one of degree 9. $H$ is not of degree 23. Let $H$ be of degree 24. Systems of imprimitivity of two, six, eight, or twelve letters are obviously impossible. No system containing $y_1$ can contain any letter of $D$ other than the letters of the constituent of degree 3 in $D$. Hence $H$ has no systems of three letters each, and of systems of four letters each, two systems are $y_1e_1e_2e_3$ and $y_2y_3y_4y_5$, which is absurd because $s_3$ does not permute these two systems. Then $H$ is primitive. If $F$ has a constituent of degree 3 or 4 in the letters $y$, new to $D$, the partitions of its degree are 8, 8, 4, 3 and 9, 8, 3, 3. The first is impossible, as then $H_1$ would have a constituent of degree 3. As above, the constituent of degree 9 should be simply isomorphic to the constituent of degree 8 of order 24. The same remarks apply to the partitions 10, 10, 3, 12, 8, 3, 9, 8, 6 and 10, 8, 5. There is one other partition, 8, 8, 7, but $H_1$ is not maximal when the partition is 8, 8, 7.

Then $H'=\{D, s_3\}$ is not transitive. The possible partitions are 16, 3, 16, 4, 12, 8, 18, 3, 12, 9, 18, 4, 14, 8, 12, 10, 20, 3, 14, 9, 20, 4, 14, 10. Here we have borne in mind that $s_3$ is similar to $s_1$ in each of the two constituents. A constituent of degree 14 is not primitive, but since it is generated by three

substitutions of order 3 and degree 9 it can have no systems of imprimitivity. Likewise a constituent of degree 12 is imprimitive, but does not respect systems of imprimitivity of three letters each ($y_1$ can not be in such a system), while it too has three generators of order 3 and degree 9. In the remaining partitions the larger constituent is generated by three substitutions of degree 12 and order 3. Assume that a constituent of degree 18 or 20 has systems of imprimitivity of two letters each. Each transposition of the substitution ($t$) of order 2 of $D$ is a system. Transform $t$ into $t'$, a substitution that involves one or two transpositions new to $t$. The product $tt'$ is not identity and is of degree 8 at most. Then the partitions 18, 3 and 18, 4 are impossible, and a transitive constituent of degree 20 has five systems of imprimitivity permuted according to a primitive group. A system of imprimitivity not composed of the four letters new to $D$ is built from a system of two letters from one, and a system of two letters from another constituent of $D$. Then the group in the systems is alternating. Now $s_3$ leaves fixed the eight letters of two systems of the constituent of degree 20, so that $t'$, the transform by $s_3$ of the substitution $t$ of order 2 of $D$ has four transpositions in common with $t$. Hence the product $tt'$, not identity, is of degree 12 at most. If the leading constituent is of degree 16, $t$ is invariant in $H'$. This is clear when we consider that the constituent in question, of order three times a power of 2, is not primitive and must accept the systems of two letters of the two constituents of $D$, even when it has systems of four letters each. If $H'$ is of order 48 it has no substitution of order 3 not in $D$. There remains then only one partition: 16, 4. The eight systems of two letters each are not permuted according to a primitive group. Then systems of four letters are permuted according to the tetraedral group. The four letters of $D$ that are fixed by $s_1$ form a system of imprimitivity. Similarly each subgroup of order 3 of $D$ fixes the four letters of a system of the constituent of degree 16. Then $s_3$ must displace at least three letters of $D$ new to $s_1$, three letters of $D$ not in $s_2$, and so on, that is to say, $s_3$ displaces letters of four distinct systems of four letters each in four cycles, which, as a matter of fact, permute only three systems.

It can be shown that if $D'=\{s_1, s'\}$ (where $s'$ is a substitution of degree 15 that displaces four letters new to $s_1$) has fewer than five transitive constituents, it contains (or is) the subgroup of the type $D$ just discussed, generated by two substitutions $s_1$ and $s_2$ similar in each transitive constituent. If $D'$ does not include the latter subgroup, it does, however, contain the subgroup $D$ of order 12 that has five constituents. Among the substitutions of $D'$ that are of degree 15 and apart from those which unite no two cycles of $s_1$, let $t$ be

a substitution that unites a minimum number of cycles of $s_1$. Both $t^2s_1t$ and $ts_1t^2$ are similar to $s_1$ in the several transitive constituents of $\{s_1, t\}$, and $\{s_1, t^2s_1t\}$ or $\{s_1, ts_1t^2\}$ has the same constituents as $\{s_1, t\}$ unless each has just five constituents, in which case $t$ displaces the six letters of two cycles of $s_1$. But then $s_1^2ts_1$ is not $t$ or $t^2$, and displaces at most three letters new to $t$.

Hence $G$ contains no subgroup $D$ of degree 19 with fewer than five constituents.

Let $D$ be the group of order 12 and degree 19. One transitive constituent is of order 3 on the letters $a_1a_2a_3$, and each of the others is an alternating group of degree 4. We may write $s_2=(b_1b_2b_4)\ldots.$, so that $s_1s_2=(b_1b_4)(b_2b_3)\ldots.$, a substitution of degree 16. Now if $s_2=(c_1c_3c_4)\ldots.$, $s_1s_2=(c_1c_2c_4)\ldots.$ Therefore,

$$s_2=a_1a_3a_2\,.\,b_1b_2b_4\,.\,c_1c_2c_4\,.\,d_1d_2d_4\,.\,e_1e_2e_4\,.$$

There exists in $G$ a substitution $s_3$, similar to $s_1$, which replaces a letter of the set $a$ by a letter of the set $b$, and by means of appropriate transformations we may write uniquely $s_3=(a_1b_1-)\ldots..$ Write

$$t=s_1^2s_2s_1=a_1a_3a_2\,.\,b_2b_3b_4\,.\,c_2c_3c_4\,.\,d_2d_3d_4\,.\,e_2e_3e_4\,.$$

Our substitution $s_3$ introduces a minimum number of new letters to $D$. Assume that $s_3$ displaces the three letters $a_1, a_2, a_3$, and in three different cycles. Then $t^2s_3t=(a_3b_1-)\ldots.$, displaces at most four letters new to $s_3$, and yet connects two cycles of $s_3$. Then $s_3$ certainly replaces an $a$ by an $a$. The transform of any one of the eight substitutions of order 3 in $D$ replaces an $a$ by $b_1$, and can not displace fewer letters new to $D$ than $s_3$ itself. Hence new letters $y$ in $s_3$ can follow only $a_2$ or $a_3$. This permits $s_3$ at most one new letter $y$. Of course, since $s_3$ unites the cycles $a_1a_2a_3$ and $b_1b_2b_3$ of $s_1$, it must displace one new letter $y$. But now, on running over the substitutions of order 3 in $D$, it is evident that $s_3$ must displace $yb_4c_4d_4e_4b_2c_2d_2e_2b_3c_3d_3e_3$, and two of the letters $c_1d_1e_1$ in addition to $a_1$ and $b_1$, which is absurd. Hence this strong condition, as holding good in all that follows:

*No two substitutions of degree* 15 *in* $G$ *have more than ten letters in common.*

Let $D=\{s_1, s_2\}$ be of degree 20. First assume it transitive. The five new letters in $s_2$ constitute one of four systems of imprimitivity permuted according to a tetraedral group. $D$ has no systems of two, four, or ten letters. A system containing one new letter $x_1$ must contain just one other new letter $x_2$, or else all the five letters $x_1x_2x_3x_4x_5$. Hence there is one and only one system of imprimitivity to which a given letter $x_1$ may belong. Then $D$ is to be looked for as a subgroup of a doubly transitive group $G$ of degree 21. All the substitutions of degree 15 in $D$ permute three of the four systems, and

leave fixed each of the letters of one system. The doubly transitive group $G$ contains a substitution $s$ of degree 15 involving a letter $y$ new to $D$. But $s$ must displace, besides $y$, four letters from each of the four systems of imprimitivity of $D$. This is absurd.

Then $D$ of degree 20 is intransitive. Constituents of degree 4 are alternating groups and transitive groups of higher degree are imprimitive groups generated by two similar substitutions of order 3 and whose systems are permuted according to an alternating group. A constituent of degree 8 is the group $\{135 . 246,\ 147 . 238\}$ of order 24. The order of a transitive constituent of degree 12 is divisible by 9. Hence the partitions 8, 4, 4, 4, 12, 4, 4 and 12, 8 are out of the question. There are three other possible partitions of the degree of $D$: 4, 4, 4, 4, 4, 8, 8, 4, 16, 4. A transitive constituent of degree 16 is of order 48 or more. The subgroup that corresponds to identity of the other constituent contains no substitution of order 3, and its order is therefore a power of 2. This leading constituent has substitutions of degree 12 and order 3, and since its order is not divisible by 9, it has no substitution of degree 15. Therefore $D$, whether the partition is 4, 4, 4, 4, 4, 8, 8, 4, or 16, 4, has exactly four conjugate subgroups of order 3. Each of them leaves fixed five letters which are displaced by the other three. We may now remark that $G$ contains no substitution of degree 15 (not a substitution of $D$) that displaces fewer than two letters new to $D$. A fact of which considerable use will be made in what follows is this: if a substitution $(t)$ similar to $s_1$ displaces or fixes the three letters of one cycle of a substitution $(s)$ similar to $s_1$, $s^2ts$ displaces at most four letters new to $t$ and is therefore $t$ itself, that is, $s$ and $t$ are commutative. The transitive group $H$ generated by $D, s_3, \ldots$, can not be of degree 22. For such a group is primitive and its subgroup $F$ contains substitutions of degree 15 displacing one letter new to $D$. Nor can $H$ be of degree 23.

Now assume that $D$ is that group which has a transitive constituent of degree 16. If $H$ is of degree 24 it is primitive, and since $H_1$ can have no constituent of degree 16, only one partition of the degree of $F$ need be proposed, 18, 5. A constituent of degree 18 is imprimitive. But $D$ does not respect systems of three letters, as the substitutions of order 5 require. Let $H$ be of degree 25, and let it first be assumed that $H$ is primitive. Of all the possible partitions of the degree of $F$ only 18, 6 and 20, 4 seem to require more than a glance. In the second partition $H_1$ has an invariant subgroup of degree 20 that corresponds to the identical substitution in the constituent of degree 4, an impossible combination.* Since a constituent of degree 6 is

* Bennett, *loc. cit.*

alternating, there are as above substitutions of order 5 and degree 15 in the imprimitive constituent of degree 18. Suppose the group $H$ to be imprimitive. A system of five letters can be chosen in but one way if it contains a given letter, and hence $H$ is not invariant in a primitive group of degree 25, nor need we pass the degree 26 in a search for a doubly transitive group of which $H$ is a subgroup. If $H$ is included in a primitive group of degree 25, the totality of substitutions of degree 15 in that group generate a primitive group which must be simply transitive and whose non-existence can be shown by the arguments used when $H$ was assumed to be primitive. The only remaining hypothesis is that $H$ is a subgroup of a doubly transitive group $G$ whose subgroup $G_1$ is imprimitive. The invariant subgroup $F$ of $G_1$, generated by all substitutions of degree 15 is imprimitive, having five systems of imprimitivity of five letters each permuted according to the icosaedral group. Its order is 1200 or a multiple thereof. The head of $F$ is of order 20 at least, and from the nature of the group $D$, which leaves fixed each of the five letters of the system formed of the letters of $F$ that are new to $D$, the head is not a simple isomorphism between its five constituents. If the head contained a substitution of order 3, each constituent would be an alternating or symmetric group. Alternating constituents, if the class of $G$ is to be maintained, can only stand in simple isomorphism to each other. Then the head contains no substitution of order 3. All the subgroups of degree 15 in $G_1$ are conjugate under the substitutions of $G_1$, which implies that the largest subgroup of $G$ in which a given subgroup of degree 15 is invariant has a transitive constituent of degree 11. But $G$ (of degree 26) can have in it no substitution of order 11. Thus the partition 16, 4 of the degree of $D$ is shown to be impossible.

The group $D$ has either five constituents of degree 4, or two transitive constituents of degree 8 and one of degree 4. Now assume that $D$ is the latter group, of order 24. Let $s_3$, with $D$, generate a transitive group. The first degree to study is 24. If $H=\{D, s_3\}$ is imprimitive, the four letters new to $D$ form a system; and the four letters of the small constituent of $D$ form a system, by which $s_3$ or $s_3^2$ replaces the former system (of new letters). Then $s_3$ displaces the three letters of a cycle of $s_1$ and is not commutative with $s_1$, hence displaces at most four letters new to $s_1$. Then $H$ is primitive. The partitions of 23, corresponding to the subgroup $F$ are 8, 8, 4, 3, 8, 8, 7, 9, 8, 6, 10, 8, 5, 10, 9, 4, 9, 9, 5. The first puts a constituent of degree 3 in $H_1$, and the last two involve a substitution of order 5 and degree 15, as do the second and third which have an alternating constituent. If the partition is 10, 8, 5, the constituent of degree 8 must include a substitution of order 5, and therefore affords a substitution of order 7 and degree 14. Let $H$ be of degree 25.

Assume $H$ to be primitive. The partitions of $F$ are 8, 8, 4, 4, 8, 8, 8, 8, 8, 5, 3, 9, 8, 4, 3, 9, 9, 6, 10, 8, 6, 10, 9, 5, 10, 10, 4, 12, 8, 4. The partitions 8, 8, 5, 3, 9, 8, 4, 3, and 10, 10, 4 impose a constituent of three or four letters on $H_1$ under impossible conditions; a constituent of degree 6 contains a substitution of order 5, thus disposing of 9, 9, 6; while the substitution of order 5 in the constituent of degree 8 (partition 10, 8, 6), and in the constituent of degree 9 (partition 10, 9, 5), leads to a substitution of order 7 in these two cases. A transitive constituent of degree 12 is imprimitive and the six systems of two letters each are permuted according to the alternating group of degree 6. This consideration reveals a substitution of order 5 and degree 15 in $F$. There remain two partitions, 8, 8, 4, 4, and 8, 8, 8. No constituent of degree 8 is primitive. The first two transitive constituents of $F$ admit only those systems of imprimitivity of two letters each, which are determined by the subgroup $D$, that is, by constituents of $D$ such as $\{123 \,.\, 456, 157 \,.\, 428\}$. Such a constituent contains no substitution of order 3 not of degree 6. A generator of $F$, not in $D$, can not fix all the letters of one of the constituents of $D$, for then that substitution would be commutative with $s_1$ and $s_2$. Then the generating substitutions of degree 15 of $F$ have just four cycles involving letters of these first two constituents, and the remaining constituent (constituents) is (are) generated by cycles of three letters. Then the third constituent is not a transitive group of degree 8. It is the direct product of two alternating groups of degree 4. A transitive constituent of degree 8, being positive, is of order 24 or 96. Either of these groups may be generated by its substitutions of order 3, so that the intransitive constituent of $F$ of degree 16 could have the factor 9 in its order only if it were the direct product of its two transitive constituents, which clearly is not possible. Let it now be assumed that $H$, of degree 25, is imprimitive. The group in the systems is the alternating group on five letters. There is only one possible system that contains one of the letters fixed by $D$, the system constituted of these five new letters. Then at once we know that neither $H$ nor a larger imprimitive group of the same degree which includes $H$ is invariant in a primitive group of degree 25, and that a primitive group of degree $n$ that includes $H$ is $n-24$ times transitive. A primitive group of degree 25, generated by substitutions of degree 15 would be disposed of by the arguments given above. We are only interested in the possible existence of a doubly transitive group $G$ of degree 26. Consider the subgroup $F$ of $G_1$, generated by substitutions of degree 15. The five systems of imprimitivity of $F$ are permuted according to an icosaedral group, and if the order of $F$ is divisible by 9, there is a substitution of order 3 in the intransitive head that permutes no systems. One of these substitutions of order 3 in the head is of

degree 15, connects two transitive constituents of $D$ and has at most three letters new to $D$, because a system of imprimitivity of $F$ can be made by two letters from each transitive constituent of degree 8, and one letter from the constituent of degree 4 or by the five letters new to $D$. Then a subgroup of order 3 is a Sylow subgroup of $G_1$, which implies a substitution of order 11 in $G$.

The subgroup $H'=\{D, s_3\}$ is intransitive. We recall that $s_3$ has at least two letters new to $D$. Among the eight substitutions of order 3 in $D$, a given letter $a_1$ of a constituent of degree 8 is in the same cycle with six of the seven remaining letters of that constituent. Therefore the transform by $s_3$ of at least one of them will, like $s_3$, connect two constituents of $D$ in one of its cycles, and may be used for $s_3$. Hence it may be assumed that $s_3$ is similar in each of the constituents of $H'$ to one of the substitutions of order 3 in $D$. The plausible partitions are 18, 4, 18, 5, 20, 4, 20, 5, 10, 12, 8, 14, 9, 14, 8, 15, 10, 14, 9, 15, 10, 15. The partitions involving constituents of degree 14, 18 and 20 yield to the arguments of page 301. A constituent of degree 12 is not primitive, but must admit four systems of imprimitivity of three letters each, from which it follows that a constituent of degree 10 is not simply isomorphic to it. The remaining partitions, 8, 15, 9, 15, 10, 15 should also be simple isomorphisms between the two constituents. Let a constituent of degree 15 be primitive. It is not multiply transitive. Then being of class 8 it is the primitive group of degree 15 and order 720, simply isomorphic to the symmetric group of that order. But this symmetric group can not be written transitively on eight, nine, or ten letters. Then a constituent of degree 15 is imprimitive, and since it is generated by substitutions of degree 9 and order 3, it has five systems of three letters each permuted according to the alternating group of degree 5. Since the second constituents are alternating groups, the partitions 8, 15 and 9, 15 are impossible. A constituent of degree 10 is imprimitive and has systems of two letters permuted according to the alternating group of order 60. Its order is not divisible by 9.

Now consider any two substitutions of degree 15, $s_1$ and $s$. If $s$ displaces more than five letters new to $s_1$ and is not commutative with $s_1$, $s^2 s_1 s$ displaces exactly five letters new to $s_1$ and, as has just been shown, does not connect two cycles of $s_1$. Then

$$s=(a_1x_1-)\,(b_1x_2-)\,(c_1x_3-)\,(d_1x_4-)\,(e_1x_5x_6),$$

say. If $s=(b_1x_2a_2)\ldots$, $s^2s_1s=(x_1b_1b)\ldots$, but $s$ can not replace $a_3$ by a $b$. Hence $s$ does not connect two cycles of $s_1$. By hypothesis the transitive constituents of $\{s_1, s\}$ are in part icosaedral. If there are one or more tetraedral constituents $\{s_1, s\}$ is a direct product and of class less than 15. *Hence if* $s$

*has just two new letters in one of its cycles, it has two new letters in every cycle, and* $\{s, s_1\}$ *is a simple isomorphism between five alternating groups of order* 60. *If any two substitutions of degree* 15 *are so related that one connects cycles of the other they are commutative, and have at most nine letters in common.*

The last of the groups $D$ is

$$\{s_1,\ s_2=a_1a_2a_4\,.\,b_1b_2b_4\,.\,c_1c_2c_4\,.\,d_1d_2d_4\,.\,e_1e_2e_4\}.$$

A substitution $s_3$ exists that connects two of the constituents of $D$, and it may be taken uniquely as

$$s_3=a_1b_1c_1\,.\,a_2b_2c_2\,.\,a_3b_3c_3\,.\,a_4b_4c_4\,.\,x_1x_2x_3\,.$$

Next, $s_4$ may be chosen with at most one letter new to $\{s_1, s_2, s_3\}$ in any cycle. Then there are just two distinct forms that $s_4$ may assume:

$$s_4=a_1d_1b_1\,.\,a_2d_2b_2\,.\,a_3d_3b_3\,.\,a_4d_4b_4\,.\,x_1y_1x_2,$$

and

$$s_4=a_1x_1a_2\,.\,b_1x_2b_2\,.\,c_1x_3c_2\,.\,d_1y_1d_2\,.\,e_1y_2e_2\,.$$

If we proceed with the second form of $s_4$, we are led at once to a substitution $s_5=a_1d—\ldots.$, so that we are at liberty to use for $s_4$ the first only, and quite similarly we have for the next step

$$s_5=a_1e_1b_1\,.\,a_2e_2b_2\,.\,a_3e_3b_3\,.\,a_4e_4b_4\,.\,x_1z_1x_2,$$

and then finally

$$s_6=a_1x_1a_2\,.\,b_1x_2b_2\,.\,c_1x_3c_2\,.\,d_1y_1d_2\,.\,e_1z_1e_2$$

closes the series of generators of a transitive group $H$ of degree 25 and order 360: the direct product of the two icosaedral groups

$$\{a_1a_2a_3\,.\,b_1b_2b_3\,.\,c_1c_2c_3\,.\,d_1d_2d_3\,.\,e_1e_2e_3,\quad a_1a_2a_4\,.\,b_1b_2b_4\,.\,c_1c_2c_4\,.\,d_1d_2d_4\,.\,e_1e_2e_4,$$
$$a_1a_2x_1\,.\,b_1b_2x_2\,.\,c_1c_2x_3\,.\,d_1d_2y_1\,.\,e_1e_2z_1\}$$

and

$$\{a_1b_1c_1\,.\,a_2b_2c_2\,.\,a_3b_3c_3\,.\,a_4b_4c_4\,.\,x_1x_2x_3,\quad a_1b_1d_1\,.\,a_2b_2d_2\,.\,a_3b_3d_3\,.\,a_4b_4d_4\,.\,x_1x_2y_1,$$
$$a_1b_1e_1\,.\,a_2b_2e_2\,.\,a_3b_3e_3\,.\,a_4b_4e_4\,.\,x_1x_2z_1\}.$$

The primitive group $G$ can contain no other substitution of degree 15, so that $H$ is an invariant subgroup, and *the degree of no primitive group of class* 15 *exceeds* 25. Now instead of the special case before us let us turn to the imprimitive group $H'$ generated by

$$s_2=a_1a_2\,.\,b_1b_2\,.\,c_1c_2\,.\ldots.\,j_1j_2\,.\,k_1k_2,$$
$$s_3=a_1a_3\,.\,b_1b_3\,.\,c_1c_3\,.\ldots.\,j_1j_3\,.\,k_1k_3,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$s_k=a_1a_k\,.\,b_1b_k\,.\,c_1c_k\,.\ldots.\,j_1j_k\,.\,k_1k_k,$$
$$s'_2=a_1b_1\,.\,a_2b_2\,.\,a_3b_3\,.\ldots.\,a_jb_j\,.\,a_kb_k,$$
$$s'_3=a_1c_1\,.\,a_2c_2\,.\,a_3c_3\,.\ldots.\,a_jc_j\,.\,a_kc_k,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$s'_k=a_1k_1\,.\,a_2k_2\,.\,a_3k_3\,.\ldots.\,a_jk_j\,.\,a_kk_k,$$

in which $H$ (after obvious changes in the lettering) is a characteristic subgroup. $H'$ is of order $(k!)^2$. A primitive group of class $2k$ is generated by $H'$ and the following substitution:

$$\begin{aligned} t = &(a_1)\,(a_2b_1)\,(a_3c_1)\,(a_4d_1)\,\ldots\,(a_ij_1)\,(a_kk_1),\\ &(b_2)\,(b_3c_2)\,(b_4d_2)\,\ldots\,(b_ij_2)\,(b_kk_2),\\ &(c_3)\,(c_4d_3)\,\ldots\,(c_ij_3)\,(c_kk_3),\\ &\ldots\ldots\ldots\ldots\ldots,\\ &(j_j)\,(j_kk_j),\\ &(k_k). \end{aligned}$$

This group is of order $2(k!)^2$ and is simply isomorphic to the intransitive group

$$(aa'\cdot bb'\cdot\ldots\cdot kk')\,(abc\ldots k)\text{ all }(a'b'c'\ldots k')\text{ all}.$$

It is the largest primitive group of degree $k^2$ in which $H'$ is invariant.* The transitive subgroups of this group in which

$$(abc\ldots k)\text{ pos }(a'b'c'\ldots k')\text{ pos}$$

is invariant are

$$\begin{aligned} &(aa'\cdot bb'\cdot cc'\cdot\ldots\cdot kk')\,(abc\ldots k)\text{ pos }(a'b'c'\ldots k')\text{ pos},\\ &(aa'\cdot bb'\cdot cc'\cdot\ldots\cdot kk')\,\{(abc\ldots k)\text{ all }(a'b'c'\ldots k')\text{ all}\}\text{ pos},\\ &(ab'ba'\cdot cc'\cdot dd'\cdot\ldots\cdot kk')\,\{(abc\ldots k)\text{ all }(a'b'c'\ldots k')\text{ all}\}\text{ pos}, \end{aligned}$$

of the orders $(k!)^2/2$, $(k!)^2$, $(k!)^2$, respectively.† Maximal subgroups of these three groups are

$$\begin{aligned} &(aa'\cdot bb'\cdot cc'\cdot\ldots\cdot kk')\,(abc\ldots j)\text{ pos }(a'b'c'\ldots j')\text{ pos},\\ &(aa'\cdot bb'\cdot cc'\cdot\ldots\cdot kk')\,\{(abc\ldots j)\text{ all }(a'b'c'\ldots j')\text{ all}\}\text{pos},\\ &(ab'ba'\cdot cc'\cdot dd'\cdot\ldots\cdot kk')\,\{(abc\ldots j)\text{ all }(a'b'c'\ldots j')\text{ all}\}\text{ pos}, \end{aligned}$$

respectively, with respect to which each can be represented as a primitive group of degree $k^2$, when $k$ is greater than 3. The first, of order $(k!)^2/2$, is generated by $H$ and $t$; the second, of order $(k!)^2$, is generated by $H$, $t$ and

$$s_2s_2' = (a_1b_2)\,(a_2b_1)\,(c_1c_2)\,\ldots\,(k_1k_2)\,(a_3b_3)\,\ldots\,(a_kb_k).$$

It contains the preceding group of order $(k!)^2/2$. The third group, also of order $(k!)^2$, is generated by $H$, $s_2s_2'$ and

$$\begin{aligned} s_2t = &(a_1b_1b_2a_2)\,(c_1b_3c_2a_3)\,(d_1b_4d_2a_4)\,\ldots\,(k_1b_kk_2a_k),\\ &(c_3)\,(c_4d_3)\,\ldots\ldots\ldots\,(c_ij_3)\,(c_kk_3),\\ &\ldots\ldots\ldots\ldots\ldots\ldots,\\ &(j_j)\,(j_kk_j),\\ &(k_k). \end{aligned}$$

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* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVIII (1906), p. 236.

† Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 299, Theorems III and IV.

It does not contain the first of the three as a subgroup. These groups are simply transitive and are of class $3k$, when $k$ is greater than 3. This last statement can readily be proved in connection with the important proposition that the group $\{H', t\}$ is the largest group on the same $k^2$ letters in which $H$ is invariant. Both statements are known to be true when $k$ is 4.* Let $G'$ be the largest group of degree $k^2$ in which $H$ is invariant. It is a simply transitive primitive group because systems of imprimitivity of $H$ with one letter in common can be chosen in only two ways. Its subgroup leaving fixed the letter $k_k$ is intransitive, and has the same two sets of intransitivity as $\{s_2, \ldots, s_j, s_2', \ldots, s_j', t\}$, the subgroup of $\{H', t\}$ that leaves $k_k$ fixed. The larger conconstituent of the latter subgroup is the primitive group of degree $(k-1)^2$ belonging to the family $\{H', t\}$. Since $\{H', t\}$ when $k$ is 4 is not contained in a larger group in which $H$ is invariant, the group $G'$ coincides with $\{H', t\}$ for all values of $k$ greater than 3. The smaller constituent in the subgroup $G_1'$ that leaves $k_k$ fixed is the imprimitive group

$$(a_k k_1 \cdot b_k k_2 \cdot \ldots \cdot j_k k_j)\,(a_k b_k \ldots . j_k) \text{ all } (k_1 k_2 \ldots . k_j) \text{ all}$$

of degree $2(k-1)$ and order $2[(k-1)!]^2$; all the substitutions in its tail are of degree $2(k-1)$. Those substitutions of $G_1'$ that are in the division of $G_1'$ that involves only positive substitutions of

$$(a_k b_k \ldots . j_k) \text{ all } (k_1 k_2 \ldots . k_j) \text{ all}$$

are of degree $3k$ at least, if they are of degree $3(k-1)$ or more in the letters of the other constituent of $G_1'$. The substitutions of $G_1'$ which are in the tail with respect to the direct product of the two symmetric groups of degree $k-1$ are of degree $5(k-1)$ at least, if they are of degree $3(k-1)$ or more in the letters of the constituent of degree $(k-1)^2$. Then by a complete induction from the groups of degree 16, the three primitive groups above are of class $3k$. What intransitive subgroups of $\{H', t\}$ contain $H$, when this group is written as a transitive group of degree $2k$? Only

$$\left.\begin{array}{l}(abc \ldots . k) \text{ all } (a'b'c' \ldots . k') \text{ all}\end{array}\right\} \text{pos},$$
$$(abc \ldots . k) \text{ all } (a'b'c' \ldots . k') \text{ pos},$$

and

$$(abc \ldots . k) \text{ pos } (a'b'c' \ldots . k') \text{ all},$$

include

$$(abc \ldots . k) \text{ pos } (a'b'c' \ldots . k') \text{ pos},$$

and they are imprimitive when represented on $k^2$ letters.

* Miller, "On the Primitive Substitution Groups of Degree 16," AMERICAN JOURNAL OF MATHEMATICS, Vol. XX (1898), p. 234. It may be noted that $G_{20}$ does not contain $G_{17}$ as a self-conjugate subgroup.

In my paper on the groups of class 12 it is stated that a certain doubly transitive group of degree 28, as well as its maximal subgroup of degree 27, seem to be new. As a matter of fact the generators there given correspond respectively to the substitutions [1, 0, 0, 0, 0, 0], [0, 1, 0, 0, 0, 0], [1, 1, 1, 1, 1, 1], [0, 0, 0, 0, 1, 0], [0, 1, 0, 0, 0, 1], [0, 0, 1, 0, 1, 1], [0, 0, 1, 1, 1, 1], as is readily seen if we put $a_1$ for the letter (0, 0, 1, 1, 0, 0). Although Jordan gives this group and its sixty-three substitutions of order 2 and degree 12 in the *Traité* of 1870, page 229, he omits it and its subgroup of order 51840 from the list of primitive groups of the first thirteen classes published in 1872.* My existence proofs on pages 258 and 259 of the paper on the "Groups of Class 12" are faulty. The present identifications are designed to replace them. The substitutions $s_1, s_2, \ldots, s_5', s_6', s_7'$, of that paper, page 259, correspond to *ab*, *ac*, *ad*, *de*, *df*, *dg*, *dh*, respectively, and if we associate with *ad* the letter $a_1$ the symmetric group of degree 8 permutes the twenty-eight binary products *ab*, *ac*, *ad*, ...., according to the substitutions of our simply transitive primitive group of degree 28 and class 12.

To make reference easier a table of the seven primitive groups of class 15 is given below. The second group is doubly and the third triply transitive; all the others are simply transitive.

| Degree | Order | Generating Substitutions |
|---|---|---|
| 16 | 80 | $G_1 = \{ab \,.\, cd \,.\, ef \,.\, gh \,.\, ij \,.\, kl \,.\, mn \,.\, op,\ boejc \,.\, dpknl \,.\, fhgim\}$ |
| 16 | 240 | $G_2 = \{G_1,\ bmn \,.\, cik \,.\, deh \,.\, flo \,.\, gpj\}$ |
| 17 | 4080 | $G_3 = \{G_2,\ aq \,.\, fk \,.\, dg \,.\, ej \,.\, il \,.\, mn \,.\, co \,.\, hp\}$ |
| 21 | 2520 | $G_4 = \{agb \,.\, chl \,.\, dim \,.\, ejn \,.\, fko,\ ahi \,.\, blm \,.\, cpd \,.\, eqs \,.\, frt,$ $ajk \,.\, bno \,.\, cqr \,.\, dst \,.\, euf\}$ |
| 25 | 7200 | $G_5 = \{H,\ a_2b_1 \,.\, a_3c_1 \,.\, a_4d_1 \,.\, a_5e_1 \,.\, b_3c_2 \,.\, b_4d_2 \,.\, b_5e_2 \,.\, c_4d_3 \,.\, c_5e_3 \,.\, d_5e_4\}$ |
| 25 | 14400 | $G_6 = \{G_5,\ a_1b_2 \,.\, a_2b_1 \,.\, c_1c_2 \,.\, d_1d_2 \,.\, e_1e_2 \,.\, a_3b_3 \,.\, a_4b_4 \,.\, a_5b_6\}$ |
| 25 | 14400 | $G_7 = \{H,\ a_1b_2 \,.\, a_2b_1 \,.\, c_1c_2 \,.\, d_1d_2 \,.\, e_1e_2 \,.\, a_3b_3 \,.\, a_4b_4 \,.\, a_5b_5,$ $a_1b_1b_2a_2 \,.\, c_1b_3c_2a_3 \,.\, d_1b_4d_2a_4 \,.\, e_1b_5e_2a_5 \,.\, c_4d_3 \,.\, c_5e_3 \,.\, d_5e_4\}$ |

In the last three groups

$$H = \{a_1a_2a_3 \,.\, b_1b_2b_3 \,.\, c_1c_2c_3 \,.\, d_1d_2d_3 \,.\, e_1e_2e_3,\quad a_1a_4a_5 \,.\, b_1b_4b_5 \,.\, c_1c_4c_5 \,.\, d_1d_4d_5 \,.\, e_1e_4e_5,$$
$$a_1b_1c_1 \,.\, a_2b_2c_2 \,.\, a_3b_3c_3 \,.\, a_4b_4c_4 \,.\, a_5b_5c_5,\quad a_1d_1e_1 \,.\, a_2d_2e_2 \,.\, a_3d_3e_3 \,.\, a_4d_4e_4 \,.\, a_5d_5e_5\}.$$

STANFORD UNIVERSITY, *September*, 1916.

* *Comptes Rendus, loc. cit.*

# *Multiple Integrals Over Infinite Fields, and the Fourier Multiple Integral.**

By Burton H. Camp.

## Introduction.

1. Multiple integrals over infinite fields have been defined in recent years in different ways by Pierpont,† Vallée-Poussin,‡ and Hardy.§ The integrals of Pierpont and of Vallée-Poussin exist only when they exist absolutely, but this is not the case with Hardy's integral. It is shown here that, for functions whose integrals over finite fields exist, these three infinite integrals are equivalent when any one of them exists absolutely. This paper is divided into three parts. In the first part (§§ 2–5) are developed relations between Hardy's integral and the iterated integrals, certain new theorems on inversion of the order of integration, and some fundamental properties of multiple infinite integrals, most of which are known for the case of integrals over finite fields but not for the infinite case. Among these last is the second theorem of the mean. Heretofore, this has not been proved, apparently, in the infinite case even for functions of one variable. The second part (§§ 6–8) deals with the behavior of integrals with respect to parameters. I am mainly concerned here with the infinite integrals, but it is necessary incidentally to extend to the case of integrals over finite fields in space of several dimensions theorems which are known only for integrals over one-dimensional fields. To save space, and because novel methods would not be introduced, the proofs of many of the statements in this part are not given. They have to do, however, with fundamental matters, including what are, in one dimension, well-known tests for termwise integration and differentiation; in the infinite case they include the so-called fundamental theorem of the integral calculus. A résumé of the third part (§§ 9–11), which contains certain applications to Fourier's double and quadruple integrals, is given at the beginning of § 9.

* Read in part before the American Mathematical Society, February 27, 1915, April 29, 1916.

† *Transactions of the American Mathematical Society*, Vol. VII (1906), p. 169.

‡ *Cours d'Analyse*, 2d ed., Vol. II, pp. 69, 70, 108, 109.

§ *Messenger of Mathematics*, Vol. XXXII (1902–3), pp. 92 ff.

FUNDAMENTAL THEORY, §§ 2–5.

2. **Elementary Theorems.** As we shall not be concerned usually with integration over finite fields, we make the assumption always that the functions used are *absolutely* $L$-integrable * in each *limited* measurable set of their domain of definition. Let $R$ denote the infinite rectangle † where $x, y \geqq 0$, and $R_{\alpha,\beta}$ the finite rectangle where $0 \leqq x \leqq \alpha$, $0 \leqq y \leqq \beta$. Let $f(x, y)$ be defined in $R$. We may then define the following integrals, when the limits in question exist:

$$J_1(f) = \lim_{\alpha,\beta=\infty} \int_{R_{\alpha,\beta}} f dx dy,$$

$$J_2(f) = \lim_{\alpha=\infty} \lim_{\beta=\infty} \int_{R_{\alpha,\beta}} f dx dy,$$

$$J_3(f) = \lim_{\beta=\infty} \lim_{\alpha=\infty} \int_{R_{\alpha,\beta}} f dx dy,$$

$$J_4(f) = \lim_{\alpha=\infty} \int_0^\alpha dx \lim_{\beta=\infty} \int_0^\beta f dy = \int_0^\infty dx \int_0^\infty f dy,$$

$$J_5(f) = \lim_{\beta=\infty} \int_0^\beta dy \lim_{\alpha=\infty} \int_0^\alpha f dx = \int_0^\infty dy \int_0^\infty f dx.$$

The integral $J_1(f)$ is Hardy's definition referred to in the introduction, $J_4$ and $J_5$ are the usual iterated integrals, $J_2$ and $J_3$ are introduced chiefly for convenience in developing the theory of $J_1$, $J_4$ and $J_5$. When it is desired to integrate a function over a larger portion or over all of the infinite two-dimensional field, we let $R_1$ denote the rectangle $(x, y \geqq 0)$, $R_2$ the rectangle $(x < 0 \leqq y)$, $R_3$ the rectangle $(x, y < 0)$, and $R_4$ the rectangle $(y < 0 \leqq x)$, and let $R = R_1 + R_2 + R_3 + R_4$, and then define the integral over $R$ as the sum of the integrals over $R_1$, $R_2$, $R_3$, and $R_4$. These definitions could be generalized slightly by using the concept of the Harnack-Lebesgue integral, but that is not worth while for my present purpose. By $\lim_{\alpha,\beta=\infty}$ is meant the double limit ‡ as $\alpha$, $\beta$ diverge to plus infinity passing through all positive values. Most of the theorems, though stated only for two dimensions, may obviously be extended to $n$ dimensions.

The two following theorems are almost immediate consequences of our definition, and the succeeding lemma is known.

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* *I. e.*, integrable in the sense of Lebesgue.

† The case of $n$ variables is not essentially different from the case of two, but the treatment would be cumbersome. In § 6, however, where finite fields only are employed, and there is no loss of simplicity, $n$ variables are used.

‡ *E. g.*, Pierpont, *Theory of Functions of Real Variables*, Vol. I (1905), p. 195.

THEOREM 1.* *A necessary and sufficient condition that $J_1(f)$ shall exist is that, for every pair of constants $h, k>0$,*

$$\lim_{\alpha, \beta=\infty} \int_r f dx dy = 0,$$

*where* $r = R_{\alpha+h, \beta+k} - R_{\alpha, \beta}$.

THEOREM 2. *If $J_i(f)$ and $J_i(g)$ exist, and $c$ is a constant, $J_i(f)+J_i(g) = J_i(f+g)$, and $J_i(cf) = cJ_i(f)$.*

LEMMA. *If for all values of $\Delta\alpha, \Delta\beta \geqq 0$, $\phi(\alpha+\Delta\alpha, \beta+\Delta\beta) \geqq \phi(\alpha, \beta)$, and $\phi(\alpha, \beta)$ is less than a fixed number $M$, then $\lim_{\alpha,\beta} \phi(\alpha, \beta)$, $\lim_\alpha \lim_\beta \phi(\alpha, \beta)$, $\lim_\beta \lim_\alpha \phi(\alpha, \beta)$ all exist and are equal and not greater than $M$.*

THEOREM 3. *If, in general,† $0 \leqq f(x, y) \leqq g(x, y)$, and if $J_i(g)$ exists, so does $J_i(f)$, and $J_i(f) \leqq J_i(g)$.*

If $i=1$, 2, or 3, this is an immediate consequence of the lemma. If $i=4$, by hypothesis there exists, except perhaps for a null set, a positive $L$-integrable function,

$$L(x) = \int_0^\infty g dy,$$

whose integral from 0 to $\alpha$ is a monotonic increasing function of $\alpha$ and has the limit $J_4(g)$. Except perhaps for a null set of $x$'s,

$$\int_0^\beta f dy = \leqq \int_0^\beta g dy,$$

and the former is a monotonic increasing function of $\beta$ whose limit $l(x)$ is less than or at most equal to $L(x)$. The integral of $l(x)$ from 0 to $\alpha$ is a monotonic increasing function of $\alpha$ and hence

$$\lim_\alpha \int_0^\alpha l(x) dx = J_4(f) \leqq \lim_\alpha \int_0^\alpha L(x) dx = J_4(g).$$

COROLLARY.‡ *The function $J_i(f)$ exists if $J_i(g)$ does, when $f(x, y) = g(x, y)$ in a part of $R$ and zero elsewhere, provided that the field is so divided that $f$ is $L$-integrable over every limited, measurable set in each part.*

THEOREM 4. *A necessary and sufficient condition that $J_i(f)$ shall exist is that there shall exist two other functions $g(x, y)$, $G(x, y)$, for both of*

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*The proof of this depends on the following theorem, well known for functions of *one* variable: A necessary and sufficient condition that the double limit of $\phi(\alpha, \beta)$ shall exist, as $\alpha, \beta$ become infinite, is that, for every pair of constants $h, k>0$, $\lim_{\alpha, \beta=\infty} [\phi(\alpha+h, \beta+k) - \phi(\alpha, \beta)] = 0$.

†*I. e.*, except perhaps at a set of points of measure zero. Such a set will be called a null set.

‡This is not necessarily true if $g$ has both signs. Indeed, this property, together with the properties in Theorem 2, would be sufficient to show that $|f|$ would be integrable, no matter what the definition of integration for $f$.

*which $J_i$ exists, such that in general $g(x, y) < f(x, y) < G(x, y)$. Then also $J_i(g) \leqq J_i(f) \leqq J_i(g)$.*

Since $0 < f-g < G-g$, $J_i(f-g) \leqq J_i(G-g)$. Then, since $J_i(g)$ exists, we have $J_i(f) \leqq J_i(G)$.

COROLLARY 1. *The same is true, with an appropriate change in the last statement, if $g \leqq f \leqq G$.*

COROLLARY 2. *The function $J_i(f)$ exists if it exists absolutely, or if $f$ is the product of two positive functions which are less than unity and for each of which $J_i$ exists, or if it is the p-th power $(p>1)$ of such a function.*

COROLLARY 3. *Under the usual conditions the mean-value theorem, $mJ_i(g) \leqq J_i(fg) \leqq MJ_i(g)$, is valid.*

COROLLARY 4. *If $J_i(f^2)$ and $J_i(g^2)$ exist, then $J_i(fg)$ exists absolutely, and the inequality of Schwarz, $J_i^2(fg) \leqq J_i(f^2) \cdot J_i(g^2)$, is valid.**

CASE I: $(i=1)$. Let $\psi=f$ and $\phi=0$, $\gamma=g$ and $\delta=0$ where $fg \geqq 0$, and let $\phi=f$ and $\psi=0$, $\delta=g$ and $\gamma=0$ where $fg<0$. The same equalities hold when both sides are squared, and by the corollary of Theorem 3 the following exist: $J_1(\psi^2)$, $J_1(\phi^2)$, $J_1(\gamma^2)$, $J_1(\delta^2)$. By the inequality for finite fields,

$$\left(\int_{R_{\alpha,\beta}} \psi\gamma\, dxdy\right)^2 \leqq \left(\int_{R_{\alpha,\beta}} \psi^2 dxdy\right)\left(\int_{R_{\alpha,\beta}} \gamma^2 dxdy\right) \leqq J_1(\psi^2) \cdot J_1(\gamma^2).$$

Since $\psi\gamma \geqq 0$, the function of $\alpha, \beta$ on the left satisfies the conditions of the lemma to Theorem 3. Therefore $J_1^2(\psi\gamma)$ exists, and so does $J_1(\psi\gamma)$. Similarly, $J_1(\phi\delta)$ exists, and since by the early equalities $|fg| = \psi\gamma - \phi\delta$, so does $J_1(|fg|)$. Now, proceeding to the limit in the relation,

$$\left(\int_{R_{\alpha,\beta}} fg\, dxdy\right)^2 \leqq \left(\int_{R_{\alpha,\beta}} f^2 dxdy\right)\left(\int_{R_{\alpha,\beta}} g^2 dxdy\right),$$

we may establish the proposition.

CASE II: $(i>1)$. This may be referred to Case I by the use of the corollary to Theorem 5 and the first part of Corollary 2, Theorem 4.

3. **The Integrals of Pierpont and Vallée-Poussin.**† In considering these definitions we shall assume as in § 2 that $f(x, y)$ is absolutely $L$-integrable in every limited measurable part of the field $R$. This is not done by Pierpont, but it is better for our present purpose to dispense with the extra element of generality which he obtains by omitting this condition. Let $R$ be defined as before, and let $D_1 < D_2 < \ldots$. be any infinite sequence of limited, measurable

* It does not follow, as in the finite case, that $f$ is integrable if $f^2$ is.

† Cf. § 1.

point sets of such a nature that each $D$ is wholly in $R$ and that every point of $R$ is in some $D$.

*Vallée-Poussin's Definition, $V(f)$.* Let $f \geqq 0$. If the following limit exists and is independent of the choice of $D$'s,

$$V(f) = \int_R f dx dy = \lim_{i=\infty} \int_{D_i} f dx dy. \tag{1}$$

Let $f$ be unrestricted as to sign. Set $f = f_1 - f_2$, where $f_1, f_2 \geqq 0$. When the integrals on the right exist,

$$V(f) = \int_R f dx dy = \int_R f_1 dx dy - \int_R f_2 dx dy. \tag{2}$$

*Pierpont's Definition, $P(f)$.* Let $f$ be unrestricted as to sign. Let the following be an infinite sequence of limited measurable sets, $E_1, E_2, \ldots$; and let $E_i$ contain every point whose distance from the origin is less than $i$; it may contain other points. When the following limit exists and is independent of the choice of $E$'s,

$$P(f) = \int_R f dx dy = \lim_{i=\infty} \int_{E_i} f dx dy. \tag{3}$$

It will be shown that this definition is equivalent to the definition:

$$P(f) = \lim_{i=\infty} \int_{D_i} f dx dy. \tag{4}$$

It is obvious that if (3) exists (4) exists and gives the same result. To prove the converse we let (4) exist. Suppose that, for a certain $E$ set, the limit in (3) should not exist, *i. e.*, suppose that the sequence

$$\int_{E_1} f dx dy, \quad \int_{E_2} f dx dy, \ldots. \tag{5}$$

should diverge. In this sequence there would exist a divergent sub-sequence

$$\int_{E'} f dx dy, \quad \int_{E''} f dx dy, \ldots. \tag{6}$$

for which $E' < E'' < \ldots$. For example, we might take $E' = E_1$, and then, since $E_1$ is limited, there would exist an $E_i$ which would include all points as near to the origin as any of those of $E_1$; let $E'' = E_i$, or some $E$ farther on in the sequence, if necessary, in order to make (6) diverge. This sub-sequence would be a $D$ sequence.

LEMMA. *If $f(x, y) \geqq 0$ and the following limit exists for one set of $D$'s, it does for every set:*

$$\lim_{i=\infty} \int_{D_i} f dx dy.$$

THEOREM 5. *(a) If* $f \geqq 0$, $V(f)=P(f)=J_1(f)$, *provided one of these exists. (b) If* $V(f)$ *or* $P(f)$ *exists, it exists absolutely, and then* $V(f)=P(f)=J_1(f)$. *(c)* $J_1(f)$ *may exist when* $V(f)$ *and* $P(f)$ *do not.*

*(a)* Since $R_{a,\beta}$ is a special case of $E_i$, if $P$ exists, so does $J_1$, and $J_1=P$. By the lemma, if $J_1$ exists, so does $V$, and $V=J_1$. By definition $V=P$.

*(b)* It is known and obvious that if $V(f)$ exists so does $V(|f|)$, and

$$V(|f|)=\lim_{i=\infty}\int_{D_i} f_1\,dxdy+\lim_{i=\infty}\int_{D_i} f_2\,dxdy,\quad V(f)=\lim_{i=\infty}\int_{D_i} f_1\,dxdy-\lim_{i=\infty}\int_{D_i} f_2\,dxdy,$$

where $|f|=f_1+f_2$, $f=f_1-f_2$. By *(a)*, then, $J_1(f_1)=V(f_1)$, $J_1(f_2)=V(f_2)$. Therefore, $J_1(f)=J_1(f_1)-J_1(f_2)=V(f)$. A similar result may be obtained for $P(f)$, since it may be shown that this integral also obeys Theorem 2. Thus *(b)* is proved if the hypothesis be that $V(f)$ exists. It only remains to show that if $P(f)$ exists so does $V(f)$. The following equation is true if two of the limits exist:

$$P(f)=\lim_{i=\infty}\int_{D_i} f\,dxdy=\lim_{i=\infty}\int_{D_i} f_1\,dxdy-\lim_{i=\infty}\int_{D_i} f_2\,dxdy.$$

If it is true, $V(f)$ exists. If it is not true, but $P(f)$ exists; since $f_1, f_2 \geqq 0$, both of the last two integrals must diverge to plus infinity for some set of $D$'s, and therefore by the lemma for all sets. On this hypothesis we will show that there exists a set of $D$'s for which the first limit does not exist. Let $R_1$ be a circle of unit radius about the origin. Let

$$k_1=\int_{R_1} f\,dxdy.$$

Since the integral of $f_1$ over $R_1$ is not infinite, and since by definition $f_1=f$ where $f \geqq 0$, there must exist outside of $R_1$ a limited set $A_1$, in which $f \geqq 0$, such that

$$\int_{A_1} f_1\,dxdy > 1-k_1.$$

Hence,

$$\int_{R_1+A_1} f\,dxdy=\int_{R_1} f\,dxdy+\int_{A_1} f\,dxdy > k_1+1-k_1=1.$$

Let $D_1=R_1+A_1$. Let $R_2$ be a circle about the origin including all of $D_1$. Let

$$k_2=\int_{R_2} f\,dxdy.$$

The integral of $f_1$ over $R_2$ is not infinite, and therefore, as before, there exists a set $A_2$ outside of $R_2$ where $f \geqq 0$ such that

$$\int_{R_2+A_2} f\,dxdy=\int_{R_2} f\,dxdy+\int_{A_2} f\,dxdy=\int_{R_2} f\,dxdy+\int_{A_2} f_1\,dxdy > 2.$$

Let $D_2=R_2+A_2$; etc.

(*c*) Hardy* has given some interesting examples of this. One which will concern us later is $J_1(\cos xy)=\pi/2$.

COROLLARY.† *If $J_i(|f|)$ exists, so does $J_j(|f|)$, and $J_i(|f|)=J_j(|f|)$, and $J_i(f)=J_j(f)$.*

Vallée-Poussin has shown that if $V(|f|)$ exists, $J_4(|f|)=J_5(|f|)=V(|f|)$, and that $J_4(f)=J_5(f)=V(f)$. By the theorem, then, the corollary is proved for the cases $i, j=1, 4, 5$. The remaining cases may easily be proved by the use of the lemma to Theorem 3.

4. **More Existence and Inversion Theorems.** THEOREM 6. *Let $J_i(f)$ exist. A necessary and sufficient condition that $J_j(f)$ shall exist is that there shall exist a function $g(x, y)$ which is in general less than $f(x, y)$ for which both J's exist.*

Necessity. Let $q(x, y)>0$, and let $J_j(q)$ exist. We may take $g=f-q$, for, since $J_j(f)$ and $J_j(q)$ exist, so does $J_j(f-q)$.

Sufficiency. Since $J_i(f-g)$ exists and $f-g\geqq 0$, $J_j(f-g)$ exists by the preceding corollary. Since now $J_j(g)$ exists, so does $J_j(f-g+g)$.

COROLLARY 1. *Let $J_i(f)$ exist. A necessary and sufficient condition that $J_j(f)=J_i(f)$ is that there shall exist a function $g(x, y)$, in general less than $f(x, y)$, for which this is true.*

$J_i(f-g)=J_j(f-g)=J_j(f)-J_j(g)$. Add the equality, $J_i(g)=J_j(g)$.

COROLLARY 2. *Let one of the iterated integrals $J_4(f)$, $J_5(f)$ exist. A necessary and sufficient condition that inversion of the order of iteration be allowable is that there shall exist a function $g(x, y)$, in general less than $f(x, y)$, for which the inversion is allowable.*

COROLLARY 3.‡ *A necessary and sufficient condition that both iterated integrals of f shall exist and be equal is that this shall be true of two functions $g(x, y)$, $G(x, y)$ for which in general $g<f<G$.*

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* *Loc. cit.*, § 1, pp. 159 ff.

† Vallée-Poussin, *loc. cit.*, Vol. II, p. 121. Bromwich (*Proceedings of the London Mathematical Society*, Vol. I (1903), p. 181, third footnote) indicates that this result follows from the lemma to Theorem 3, but really only the last part, in which it is said that $J_1=J_2=J_3$, follows from that lemma. Indeed, if it did follow also that $J_1=J_4=J_5$, as he states, his result would antedate its discovery for the finite case by four years (see also Hobson, same *Proceedings*, Vol. VIII (1910), p. 22).

I am omitting from this paper the subject of transformation of variables in these integrals. It is true, however, that, if any $J_i$ exists absolutely, it may be transformed into an equivalent double, or iterated, integral over the unit square. Another proof of the corollary may be devised by the use of transformations of this sort.

‡ Cf. W. H. Young, same *Proceedings*, Vol. IX (1910–11), p. 323. Young's theorem gives only sufficient criteria that inversion of the order of iteration be permissible, and involves the additional hypothesis that this inversion be permissible for integrals of $g$ and $G$ over one-way infinite fields.

COROLLARY 4.* *(a) Inversion of the order of iteration of $f$ over the one-way infinite field $R_\alpha(0 \leqq x \leqq \alpha, 0 \leqq y)$ is allowable if one of the iterated integrals of $|f|$ exists. (b) If one of the iterated integrals of $f$ over $R_\alpha$ exists, a necessary and sufficient condition that inversion be allowable is that there shall exist a function $g(x, y)$, in general less than $f(x, y)$, for which it is allowable. (c) A necessary and sufficient condition that both iterated integrals over $R_\alpha$ shall exist and be equal is that there shall exist two functions $g(x, y)$, $G(x, y)$ for which this is so and for which in general $g<f<G$.*

COROLLARY 5. *The theorem and previous corollaries are true if we write, instead of $g<f(f<G)$, $g \leqq f(f \leqq G)$, or if, in the theorem and first two corollaries, we substitute $G>f$ for $g<f$.*

THEOREM 7. *(a) If $J_1(f)$ exists, and $J_2(f)$ exists over $R_\alpha$ for sufficiently large values of $\alpha$ (say $\alpha>C$) then $J_2(f)=J_1(f)$; similar statements may be made for $J_3$ and the one-way infinite field $R_\beta(0 \leqq y \leqq \beta, 0 \leqq x)$. (b) Let inversion† of the order of iteration over $R_\alpha(\alpha>C)$ be allowable; then $J_2(f)=J_4(f)$ if either exists; and similar statements may be made for $J_3$ and $J_5$ and $R_\beta$.*

COROLLARY. *Sufficient conditions that all the $J$'s shall exist and be equal are that inversion be allowable over both $R_\alpha$ and $R_\beta(\alpha, \beta>C)$, and that $J_1$ and one of each of the pairs $(J_2, J_4)$, $(J_3, J_5)$ shall exist. Then also all the $J$'s are equal.*

The proofs of the theorems and corollaries of this section are simple.

5. **Second Theorem of the Mean.**‡ THEOREM 8. *Let $f(x, y)$ be limited in $R$ and $L$-integrable over every finite measurable set, and let $g(x, y)$ be absolutely $L$-integrable in $R$. Then, if $l \leqq f(x, y) \leqq L$, there exists a $k$ such that*

$$J=\int_R f(x, y)g(x, y)dxdy=l\int_{R-A_k} gdxdy+L\int_{A_k} gdxdy;$$

*where, if $f$ has the property that the set of points $E_k$, where $f=k$, is always null, $A_k$ is defined as the set where $f>k$; and otherwise $A_k$ is the set where*

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* Parts (a) and (c) are known. See also note to Theorem 7 (b). By the existence of the integral of $f$ over $R_\alpha$ we mean the existence over $R$ of the integral of the function which equals $f$ in $R_\alpha$ and zero elsewhere.

† For criteria for this condition we have, not only Theorem 6, Corollary 4, but also the usual condition that, assuming $|f|$ $L$-integrable over finite fields, termwise integration with respect to $x$ of the $\beta$ sequence—viz: the integral of $f$ with respect to $y$ from 0 to $\beta$—be allowable in $(0, \alpha)$. This assumes implicitly the existence of one iterated integral.

‡ Proved for the case where $R$ is finite by Lebesgue, *Annales de l'École Normale*, (3) Vol. XXVII (1910), p. 444; proved in a slightly more general form by the author, *Mathematische Annalen*, Vol. LXXV (1914), p. 285. There will be used in this section much of the notation and some of the results of the latter paper.

*$f>k$, plus perhaps certain points where $f=k$. In the second case the set $A_k$ may be defined by the use of an associated function $q(x, y)$.*

CASE 1: $(l=0, \hat{E}_k=0)$.* Consider a monotonic sequence of circles whose union† is $R: R_1<R_2<\ldots\ldots$ Let $A_k$ be the set of points where $f>k$. Let $A'_{k_i}$ refer to the parts of $A_{k_i}$ in $R_i$. By the finite case there exists for each $R_i$ a $k_i$ such that

$$\int_{R_i} fg=L\int_{A'_{k_i}} g. \tag{1}$$

By Theorem 4, $fg$ is absolutely $L$-integrable in $R$, and by Theorem 5

$$J=\int_R fg=\lim_{i=\infty}\int_{R_i} fg=L\lim_{i=\infty}\int_{A'_{k_i}} g; \tag{2}$$

so the last limit exists. It only remains to show that there is a definite $k$ and $A_k$ for which

$$\lim_{i=\infty}\int_{A'_{k_i}} g=\int_{A_k} g. \tag{3}$$

The left-hand side may be considered as the limit of any infinite sub-sequence selected from the sequence

$$\int_{A'_{k_1}} g, \quad \int_{A'_{k_2}} g, \ldots\ldots \tag{4}$$

Suppose the sub-sequence to be the terms corresponding to $i=a_1<a_2<\ldots\ldots$, where these numbers are so chosen that, if $k$ be taken as the lower limit of the sequence $k_1, k_2, \ldots\ldots$, it is also the limit of either (5) or (6),

$$k_{a_1}\geqq k_{a_2}\geqq\ldots\ldots, \tag{5}$$

$$k_{a_1}\leqq k_{a_2}\leqq\ldots\ldots. \tag{6}$$

To be assured that this is possible we first note that whenever the points $k_1, k_2, \ldots\ldots$ constitute an infinite point set, there exists a lower limit, for the set is limited $(0\leqq k_i\leqq L)$. Whenever the points do not constitute an infinite point set, certain of these numbers are repeated an infinite number of times; let $k$ be the least of these numbers. It is immaterial to the proof whether (5) or (6) holds. Assume (5); and to avoid the additional subscripts let it be assumed that (4) *is* the sequence corresponding to (5), *i. e.*, that $k$ is the limit of $k_i$, and that

$$k_1\geqq k_2\geqq\ldots\ldots. \tag{7}$$

Then

$$A'_{k_1}\leqq A'_{k_2}\leqq\ldots\ldots. \tag{8}$$

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* By $\hat{E}$ is meant the measure of $E$, a concept which may be defined in the infinite case as the integral over $E$ of unity.

† By "union" of $(R_i)$ is meant the totality of points each of which belongs to some $R_i$.

The limit of this sequence may be an infinite field. For $\varepsilon>0$ prescribed, let $R_a$ be so large that

$$\int_{R-R_a} |g| < \varepsilon. \tag{9}$$

Let

$$A''_{k_1} \leqq A''_{k_2} \leqq \ldots. \tag{10}$$

refer to the portions of $A'_{k_1}$, etc., in $R_a$. If $i<\alpha$, $A''_{k_i}$ may not refer to all of the points of $R_a$ where $f>k_i$, for $A'_{k_i}$ may not include them all; but if $i \geqq \alpha$, $A''_{k_i}$ does include all those points, and only those. $A''_k$ is the set in $R_a$ where $f>k$; and so, by the reasoning of my *Annalen* paper,* $A''_k$ is the union of $A''_{k_i}$ and $\lim_{i=\infty} \hat{A}''_{k_i}=\hat{A}''_k$. Thus, on account of the absolute continuity of the integral of $g$ in $R_a$, there exists an $i_\epsilon>\alpha$ and so large that if $i>i_\epsilon$ $(\hat{A}''_k-\hat{A}''_{k_i})$ is sufficiently small to make

$$\int_{A''_k-A''_{k_i}} |g| < \varepsilon. \tag{11}$$

Now the set $(A_k-A'_{k_i})$ is made up of two parts, $(A''_k-A''_{k_i})$ and $B$, where $B$ is some set outside of $R_a$. Therefore, by (9) and (11),

$$\left| \int_{A_k-A'_{k_i}} g \right| < \int_{A''_k-A''_{k_i}} |g| + \int_B |g| < 2\varepsilon,$$

which proves (3), and hence the proposition.

CASE 2: $(l=0,\ \hat{E}_k \geqq 0)$. The only new difficulty here is with inequality (11). It is obvious that $(\hat{A}''_k-\hat{A}''_{k_i})$ may not become small at pleasure because of the non-null sets that may be included. Therefore a method must be found of dividing up these non-null $E_k$ sets. The measures of some of them may be infinite. We first note that there are only an enumerable number of values of $k$, say $k_1, k_2, \ldots$, where $\hat{E}_k>0$. This follows from the fact that, if $A_k$ be, as before, the set where $f>k$, $\hat{A}_k$ is a monotonic decreasing function of $k$, and $k_i$ is a point of discontinuity of $\hat{A}_k$. It is known that such points are at most enumerably infinite for monotonic functions. We now define, corresponding to the parameter $\lambda(0 \leqq \lambda \leqq L+1)$, a family of point sets $\Omega_\lambda$, and a limited function $q(x, y)$ which is associated in a certain way with $f(x, y)$, and which is constant at most in null sets, and has the property that the set of points where $q>\lambda$ is $\Omega_\lambda$. Let $c_i$ be any positive number such that

$$c_1+c_2+\ldots.=1. \tag{12}$$

* *Loc. cit.*, p. 275, Lemma 2.

Let $q(x, y)$ be defined as the limit of the monotonic sequence,

$$f(x, y) \leqq q_1(x, y) \leqq q_2(x, y) \leqq \ldots . \leqq f(x, y)+1, \tag{13}$$

where $q_1, q_2, \ldots .$ are defined as follows: at points where $f<k_1$, let $q_1$ equal $f$; at points where $f=k_1$, let $q_1$ equal $c_1e^{-x^2-y^2}+f$; and at points where $f>k_1$, let $q_1$ equal $f+c_1$. Here the function $c_1e^{-x^2-y^2}+k_1$ has the properties that it is constant only in null sets, that it lies between $k_1$ and $k_1+c_1$, and that the set of points where it is greater than any number between these limits is a circle. Other definitions of $q_1$ might obviously be chosen, resulting in other associated functions $q$. Next, let $q_2$ be equal to $q_1$ where $f<k_2$, equal to $c_2e^{-x^2-y^2}+q_1$ where $f=k_2$, and equal to $q_1+c_2$ where $f>k_2$, etc. Let $\Omega_\lambda$ be the set of points where $q(x, y)>\lambda (0 \leqq \lambda \leqq L+1)$. The proof that $q$ is constant at most in null sets is omitted because the fact seems obvious.

It will now be shown that each $\Omega_\lambda$ is divisible into two parts; one is part or all of the set where $f$ equals a certain corresponding $k_\lambda$, and the other is the whole of the set where $f>k_\lambda$. Consider the set $A_k$ where $f>k$, and suppose the $k_i$'s (7) which are not greater than $k$ to be denoted by $\alpha_i$'s and their corresponding $c_i$'s by $s_i$'s, thus:

$$\alpha_1, \alpha_2, \ldots . \leqq k;\ s_1, s_2, \ldots .$$

In $A_k$,

$$q \geqq f+s_1+s_2+\ldots . > k+s_1+s_2+\ldots . \tag{14}$$

For example, if $k_1=\alpha_1$, $k_2>k$, $k_3=\alpha_2$, $\ldots .$; in $A_k$, $q_1=f+s_1$, $q_2 \geqq q_1=f+s_1$, $q_3=q_2+s_2 \geqq f+s_1+s_2$, etc., and finally $q_1 \leqq q_2 \leqq \ldots . \leqq q$. In $R-A_k$,

$$q \leqq f+s_1+s_2+\ldots . \leqq k+s_1+s_2+\ldots ., \tag{15}$$

for here, for every $j$, $q_j \leqq f+s_1+s_2+\ldots .$, and finally $q=\lim_j q_j$. Thus, if $\lambda=k+s_1+s_2+\ldots .$, $q>\lambda$ in $A_k$ and $q \leqq \lambda$ in $R-A_k$, and so in this case $\Omega_\lambda=A_k$. If now it could be shown that to every $\lambda$ there corresponds a $k$ such that $\lambda=k+s_1+s_2+\ldots .$, the part of our proposition considered in this paragraph would be established, but this is not always true. It can, however, be proved except for an enumerable number of intervals where

$$\lambda_{k_i}-c_i<\lambda \leqq \lambda_{k_i}. \tag{16}$$

For $s_1+s_2+\ldots .$ is a monotonic increasing function of $k$, and therefore, considered as a function of $k$, $\lambda=k+s_1+s_2+\ldots .$ is always increasing, and is defined for each $k$ in $(0, L)$. Moreover, it is always continuous on the right, for, for any $\bar{k}$ which is not a $k_i$, there exists a $\delta$ so that the interval $(k, k+\delta)$ does not contain any of the points $k_1, k_2, \ldots ., k_n$ ($n$ finite). Therefore,

$$|\lambda(\bar{k}+\delta)-\lambda(\bar{k})| < |\delta| + \sum_n^\infty c_i < \varepsilon,$$

if $n$ and $\delta$ are chosen properly. This is true for $\delta>0$ even if $\overline{k}$ is a $k_i$, but in this case every interval $(\overline{k}, \overline{k}+\delta)$, $\delta<0$, would include $k_i$, and hence at such points $\lambda$ would have a discontinuity on the left equal to $c_i$. The correspondence desired is proved. We proceed to consider values of $\lambda$ of (16). First, all points of $A_{k_i}$ are in $\Omega_\lambda$, for, if $f>k_i$, $q>\lambda_{k_i}=k_i+s_1+s_2+\ldots.$ $(c_i=s_1)$, that is $q>\lambda$. Secondly, $\Omega_\lambda$ contains no other points, except some where $f=k_i$; for, if $P$ is a point where $f(P)<k_i$, let $f(P)=k_i-\gamma$. To this value of $k$ there is a corresponding $\overline{\lambda}=k_i-\gamma+\sigma_1+\sigma_2+\ldots.$, where $\sigma_1, \sigma_2, \ldots.$ are the $s_i$'s for $k_i-\gamma$; their sum is less than $s_1+s_2+\ldots.$ by at least $c_i=s_1$. By (15)

$$q(P)\leqq k_i-\gamma+\sigma_1+\sigma_2+\ldots.\leqq k_i-\gamma+s_1+s_2+\ldots.-s_1<\lambda_{k_i}-c_i<\lambda,$$

and therefore $P$ is not in $\Omega_\lambda$. Thus, in every case there is a $k_\lambda$ corresponding to $\lambda$ for which the conditions are satisfied.

It may now be shown that

$$J=L\int_{\Omega_\lambda} g.$$

As in Case 1 (1),

$$\int_{R_i} fg=L\int_{\Omega'_{\lambda_i}} g,$$

where $\Omega'_{\lambda_i}$ is the part of $\Omega_{\lambda_i}$ in $R_i$. For $q$ is for $R_i$ precisely an associated function in the sense of my paper for the case when $R$ is limited. The rest of the reasoning of Case 1 now holds, *mutatis mutandis.*

CASE 3: $(l\neq 0)$. We may obviously consider Case 1 as a special case of Case 2, where $q=f$. By Case 2,

$$\int_R [f(x,y)-l]g(x,y)=(L-l)\int_{\Omega_\lambda} g=L\int_{\Omega_\lambda} g-l\int_R g+l\int_{R-\Omega_\lambda} g.$$

Add

$$\int_R lg=l\int_R g.$$

COROLLARY. *If $f(x)$ is monotonic decreasing in $(a,\infty)$ and $l\leqq f(x)\leqq L$, and if $g(x)$ is absolutely L-integrable in the same interval, there exists a $\lambda(a\leqq\lambda\leqq\infty)$ such that*

$$\int_a^\infty fg=l\int_\lambda^\infty g+L\int_a^\lambda g.$$

It is to be noted also that the theorem gives even for one dimension a much more general statement than this corollary.

INTEGRALS CONTAINING PARAMETERS, §§ 6–8.

6. **Integrals over Finite Fields.*** As a part of my discussion of integrals containing parameters I intend to extend to integrals over infinite fields certain theorems of W. H. Young† on integrals of functions of one variable. It is necessary first to make the extension to $n$ dimensions for finite fields, and in this section I shall cite certain of these results. They may be deduced more simply from Vitali's test‡ of equi-absolute continuity, than by extending to $n$ dimensions Young's idea of upper (lower) semi-continuity, but I will not give the proofs. In passing, however, I call attention to one theorem (11) which is not related to Young's, but which is a generalization of a very useful theorem on integrals not containing parameters.

THEOREM 9. *Let $A$ be a limited measurable point set in space of $n$ dimensions. Let $\lim_{n,m} f_{n,m}(x, y, \ldots) = f(x, y, \ldots)$ in $A$, in general. The necessary and sufficient condition that*

$$\lim_{n,m=\infty} \int_A f_{nm} dA = \int_A f dA$$

*is that there shall exist two functions $g_{n,m}(x, y, \ldots)$, $G_{n,m}(x, y, \ldots)$ whose limits are $g(x, y, \ldots)$ and $G(x, y, \ldots)$, respectively, for which this is true, and such that, in general, $g_{n,m} < f_{n,m} < G_{n,m}$.*

COROLLARY 1. *An analogous theorem is true for iterated integrals.*

COROLLARY 2. *If $f_{n,m}(x, y, \ldots) \geqq 0$, and its limit is $f(x, y, \ldots)$, and if also the limit of its integral over $A$ is the integral of $f$ over $A$, then a similar statement is true for the limit of its integral over any measurable part of $A$.*

THEOREM 10. *If the limit of a monotonic sequence of integrable functions $0 \leqq f_1(x, y, \ldots) \leqq f_2(x, y, \ldots) \leqq \ldots$, is integrable in $A$, the sequence is integrable termwise.*

THEOREM 11.§ *If in $A$ $f(x, y, \ldots)$ is the limit of a function $f_{n,m}(x, y, \ldots)$ which is nowhere negative, and the limit of whose integral is zero, then in general $f = 0$.*

The proof is immediate if it can be shown that the integral of $f$ over $A$ exists, for in that case the integral of $f_{n,m}$ is obviously equi-absolutely con-

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*It is not assumed in this section, as it is elsewhere, that the functions in question have integrals over finite measurable fields. The case of $n$ parameters is not essentially different from that of two.

† *Loc. cit.*, pp. 315 ff.

‡ *Rendiconti del Circolo di Palermo*, Vol. XXIII (1907) pp. 137 ff. Vitali enunciated his test only for one dimension, but his method is valid for $n$ dimensions.

§ This is a generalization of a well-known theorem. Cf. Pierpont, *The Theory of Functions of Real Variables*, Vol. II (1911), p. 385, § 402 (2).

tinuous and approaches the integral of $f$ as a limit. Thus the integral of $f$ is zero, and the theorem of which this is a generalization applies. Suppose the integral of $f$ over $A$ did not exist. We could then let $A_k$ be the set where $f>k$, and set $\phi_k$ equal to zero in $A_k$ and to $f$ in $(A-A_k)$. Then the integral of $\phi_k$ over $A$ would diverge to plus infinity as $k$ became infinite. This would be impossible, for suppose that for each $k$ $\phi_{n,m;k}$ equals 0 in $A_k$ and $f_{n,m}$ elsewhere. Then for each $k$ $\phi_{n,m;k}$ would satisfy the conditions of the theorem and $\phi_k$ would be integrable. Therefore the integral of $\phi_k$ over $A$ would equal zero for each $k$.

7. **Continuity.** LEMMA 1. *If* $0\leqq f(x,y)\leqq g(x,y)$, *and* $J_i(g)$ *exists, then*

$$0\leqq J_i(f)-\int_{R_{\alpha,\beta}} f\,dxdy\leqq J_i(g)-\int_{R_{\alpha,\beta}} g\,dxdy.$$

LEMMA 2. *If for each* $(n,m)$ *the function* $f_{n,m}(x,y)$ *satisfies the conditions imposed on* $f$ *in Lemma* 1, *and if* $f(x,y)=lim_{n,m}f_{n,m}(x,y)$ *in general, then* $J_i(f_{n,m})$ *exists and its limit is* $J_i(f)$.

Obviously $0\leqq f\leqq g$, and so by Theorem 3 $J_i(f_{n,m})$ and $J_i(f)$ exist. For an arbitrary $\varepsilon>0$, let $\alpha$, $\beta$ be chosen so large that

$$J_i(g)-\int_{R_{\alpha,\beta}} g<\varepsilon.$$

By Lemma 1, then,

$$0\leqq J_i(f)\quad-\int_{R_{\alpha,\beta}} f\leqq J_i(g)\quad-\int_{R_{\alpha,\beta}} g<\varepsilon, \tag{1}$$

$$0\leqq J_i(f_{n,m})-\int_{R_{\alpha,\beta}} f_{n,m}\leqq J_i(g)-\int_{R_{\alpha,\beta}} g<\varepsilon. \tag{2}$$

By Theorem 9 there exists for these fixed values of $\alpha$, $\beta$ an $n_\varepsilon$ so large that, if $n, m>n_\varepsilon$,

$$\left|\int_{R_{\alpha,\beta}} f-\int_{R_{\alpha,\beta}} f_{n,m}\right|<\varepsilon. \tag{3}$$

From (1), (2), and (3),

$$|J_i(f)-J_i(f_{n,m})|<3\varepsilon,\qquad (n,m>n_\varepsilon).$$

LEMMA 3. *Except perhaps for a null set in* $R$, *let* $g_{n,m}(x,y)\geqq 0$, $lim_{n,m}g_{n,m}(x,y)=g(x,y)$, *and* $lim_{n,m}J_i(g_{n,m})=J_i(g)$. *Then, for all values of* $\alpha$, $\beta$ *uniformly,*

$$\lim_{n,m}\int_{R_{\alpha,\beta}} g_{n,m}\,dxdy=\int_{R_{\alpha,\beta}} g\,dxdy.$$

Let $\phi_{n,m}=g_{n,m}$ and $\psi_{n,m}=g$ where $g_{n,m}\geqq g$, and let $\phi_{n,m}=g$ and $\psi_{n,m}=g_{n,m}$ where $g_{n,m}<g$. Then $0\leqq\psi_{n,m}<\phi_{n,m}$,

$$0\leqq\phi_{n,m}-\psi_{n,m}=|g_{n,m}-g|, \tag{1}$$

$\lim_{n,m}\phi_{n,m}=\lim_{n,m}\psi_{n,m}=g$, and $0\leqq\psi_{n,m}\leqq g$. By Lemma 2, then,

$$\lim_{n,m}J_i(\psi_{n,m})=J_i(g). \tag{2}$$

By definition, $\phi_{n,m}+\psi_{n,m}=g_{n,m}+g$, and hence $\lim_{n,m}J_i(\phi_{n,m}+\psi_{n,m})=2J_i(g)$. This, with (1) and (2), proves that $\lim_{n,m}J_i(|g_{n,m}-g|)=0$. Now, uniformly in $\alpha, \beta$,

$$\left|\int_{R_{\alpha,\beta}}(g_{n,m}-g)\right|\leqq\int_{R_{\alpha,\beta}}|g_{n,m}-g|\leqq J_1(|g_{n,m}-g|).$$

LEMMA 4. *Except perhaps for a null set in R, let* $0\leqq f_{n,m}(x,y)\leqq g_{n,m}(x,y)$, *let* $\lim_{n,m}f_{n,m}(x,y)=f(x,y)$, *and* $\lim_{n,m}g_{n,m}(x,y)=g(x,y)$, *and* $\lim_{n,m}J_i(g_{n,m})=J_i(g)$. *Then* $J_i(f_{n,m})$ *exists and its limit is* $J_i(f)$.

Since obviously $0\leqq f\leqq g$, we have, by Lemma 1, for all values of $n, m$; $\alpha, \beta$,

$$0\leqq J_i(f)\quad-\int_{R_{\alpha,\beta}}f\quad\leqq J_i(g)\quad-\int_{R_{\alpha,\beta}}g, \tag{1}$$

$$0\leqq J_i(f_{n,m})-\int_{R_{\alpha,\beta}}f_{n,m}\leqq J_i(g_{n,m})-\int_{R_{\alpha,\beta}}g_{n,m}. \tag{2}$$

For each $\varepsilon>0$ there exists a number $n_\varepsilon$ so that, by hypothesis and by Lemma 3, for all values of $n, m>n_\varepsilon$ and all values of $\alpha, \beta$ uniformly,

$$|J_i(g_{n,m})-J_i(g)|,\quad\left|\int_{R_{\alpha,\beta}}g-\int_{R_{\alpha,\beta}}g_{n,m}\right|<\varepsilon, \tag{3}$$

and so

$$\left|\left(J_i(g_{n,m})-\int_{R_{\alpha,\beta}}g_{n,m}\right)-\left(J_i(g)-\int_{R_{\alpha,\beta}}g\right)\right|<2\varepsilon. \tag{4}$$

But, if $\alpha, \beta>$ some number $\alpha_\varepsilon$,

$$\left|J_i(g)-\int_{R_{\alpha,\beta}}g\right|<\varepsilon, \tag{5}$$

and, therefore, by (3) and (5),

$$\left|J_i(g_{n,m})-\int_{R_{\alpha,\beta}}g_{n,m}\right|<3\varepsilon,\qquad(n, m>n_\varepsilon;\alpha, \beta>\alpha_\varepsilon).$$

This, with (1) and (2), shows that

$$\left|J_i(f)-J_i(f_{n,m})+\int_{R_{\alpha,\beta}}f_{n,m}-\int_{R_{\alpha,\beta}}f\right|<4\varepsilon(n, m>n_\varepsilon;\alpha, \beta>\alpha_\varepsilon). \tag{6}$$

Now let $\alpha, \beta$ be held fast at some values greater than $\alpha_\epsilon$. There exists a number $n'_\epsilon$ so that

$$\left| \int_{R_{\alpha,\beta}} f_{n,m} - \int_{R_{\alpha,\beta}} f \right| < \epsilon, \qquad (n, m > n'_\epsilon), \tag{7}$$

for, by Lemma 3, and since $0 \leqq f_{n,m} \leqq g_{n,m}$, Theorem 9 applies. Thus (6) and (7) prove the lemma.

THEOREM 12. *Let $\lim_{n,m} f_{n,m}(x, y) = f(x, y)$ in general. In order that the following integrals shall exist and that*

$$\lim_{n,m} J_i(f_{n,m}) = J_i(f),$$

*it is necessary and sufficient that there shall exist two convergent sequences of functions, $\lim_{n,m} g_{n,m}(x, y) = g(x, y)$, $\lim_{n,m} G_{n,m}(x, y) = G(x, y)$, for which this is true, such that in general $g_{n,m}(x, y) < f_{n,m}(x, y) < G_{n,m}(x, y)$.*

To prove the necessary part we may take $g_{n,m} = f_{n,m} - 1/(mnx^2y^2)$, $G_{n,m} = f_{n,m} + 1/(mnx^2y^2)$. To prove the sufficient part we note that $0 \leqq f_{n,m} - g_{n,m} \leqq G_{n,m} - g_{n,m}$, and that $\lim_{n,m} J_i(G_{n,m} - g_{n,m}) = J_i(G - g)$, and apply Lemma 4.

COROLLARY 1. *Let $\lim_{n,m} f_{n,m}(x, y) = f(x, y)$ in general. Any one of the following is sufficient to make $\lim_{n,m} J_i(f_{n,m}) = J_i(f)$:*

*(a) $g(x, y) \leqq f_{n,m}(x, y) \leqq G(x, y)$, and $J_i(g)$, $J_i(G)$ shall exist;*

*(b) $f(x, y)$ shall be the limit of a monotonic sequence,*

$$f_1(x, y) \leqq f_2(x, y) \leqq \ldots,$$

*and $J_i(f)$ and $J_i(f_1)$ shall exist;*

*(c) $\lim_{n,m} J_i(|f_{n,m}|) = J_i(|f|)$;*

*(d) $|f_{n,m}(x, y)| \leqq G(x, y)$ and $J_i(G)$ shall exist;*

*(e) $f_{n,m}(x, y) = h_{n,m}(x, y) g_{n,m}(x, y)$, where $0 \leqq h_{n,m}, g_{n,m} \leqq 1$, and one of the convergent sequences $h_{n,m}$ and $g_{n,m}$ shall be integrable termwise in $R$;*

*(f) $f_{n,m} = (g_{n,m})^p$, $0 \leqq g_{n,m} \leqq 1 < p$, and the sequence $g_{n,m}$ shall be integrable termwise in $R$.*

COROLLARY 2.* *The double series of which the general term is $u_{n,m}(x, y)$ is absolutely convergent and absolutely integrable termwise in $R$ if there exists a never negative function $v(x, y)$ which is integrable in $R$ and a convergent double series of positive terms of the type $a_{n,m}$ such that $|u_{n,m}(x, y)| \leqq a_{n,m} v(x, y)$.*

For, if $s_{n,m}$ refer to the sum of the absolute values of the $nm$ terms of the $u$ series, and $\sigma_{n,m}$ to the sum of the corresponding terms of the $a$ series, $0 \leqq s_{n,m} \leqq \sigma_{n,m} v(x, y) \leqq \lim_{n,m} \sigma_{n,m} v(x, y)$, and (*d*) of Corollary 1 applies.

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*Cf. W. H. Young, *loc. cit.*, p. 324. The same statement holds good if $R$ is a finite field.

8. **Differentiation and Integration.** The next theorem may be derived from what precedes in much the same manner as may be proved the theorem * of which it is a generalization.

THEOREM 13. *Let* $g_n(x,y) \leqq f_n(x,y) \leqq G_n(x,y)$, *and* $f_n - g_n$, $G_n - f_n$ *be monotonic increasing functions of* $n$ *at each point* $(x,y)$, *and let the derivative of* $J_i(g_n) = J_i(dg_n/dn)$, *and the derivative of* $J_i(G_n) = J_i(dG_n/dn)$. *Then the derivative with respect to* $n$ *of* $J_i(f_n) = J_i(df_n/dn)$ *wherever*† $df_n/dn$ *exists. The same is true if the right (left) hand derivative be used throughout instead of the derivative.*

The following lemma may be proved by the use of the theorem of Young analogous to Theorem 9, and the theorem of Fubini analogous to the corollary of Theorem 5.

LEMMA. *Let the interval* $(a \leqq n \leqq b)$ *be finite. Sufficient conditions that the following exist and be equal,*

$$\int_a^b J_1[g(x,y;n)]\,dn, \quad J_1\left[\int_a^b g(x,y;n)\,dn\right],$$

*are that* (1°) *the integral of* $g$ *with respect to* $x$ *and* $y$ *over* $R_{\alpha,\beta}$, *for all values of* $\alpha, \beta > \alpha_0$, *be absolutely less than an integrable function of* $n$, (2°) $J_1(g)$ *exist for each* $n$, *and* $g$ *be absolutely integrable with respect to* $x$, $y$, *and* $n$ *in the parallelopiped,* $0 \leqq x \leqq \alpha$, $0 \leqq y \leqq \beta$, $a \leqq n \leqq b$.

THEOREM 14. *Sufficient conditions that the equation,*

$$\int_a^n J_1\left(\frac{df}{dn}\right)dn = J_1(f_n) - J_1(f_a),$$

*be valid in* $(a \leqq n \leqq b)$ *are that* $J_1(f)$ *exist at one point of the interval* $(a,b)$, *and that* $df/dn$ *exist and satisfy the conditions imposed on* $g$ *in the lemma.*

By the lemma

$$\int_a^n J_1\left(\frac{df}{dn}\right)dn = J_1\int_a^n \frac{df}{dn}\,dn.$$

By a known theorem due to Vallée–Poussin,‡ this equals $J_1(f_n) - J_1(f_a)$.

COROLLARY.§ *Under the same conditions the derivative, with respect to* $n$, *of* $J_1(f_n)$ *equals* $J_1$ *of the derivative of* $f_n$, (*a*) *in general in* $(a,b)$ *including all*

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* See preceding note, same reference, p. 321.

† More generally, for each value of $n$ for which, except perhaps for a null set in the $xy$-plane, $df/dn$ exists.

‡ *Loc. cit.*, Vol. I, p. 263.

§ See reference of previous note, pp. 267, 268.

*points where the derivative of $J_1(f_n)$ exists, and (b) at all points of $(a, b)$ provided $J_1(df/dn)$ is continuous in $(a, b)$.*

THE FOURIER INTEGRALS, §§ 9–11.

9. **Introduction.** The iterated integral

$$J_4=\frac{1}{\pi}\int_0^\infty du\int_{-\infty}^\infty f(t)\cos u(t-x)\,dt$$

is sometimes called the Fourier double integral. We shall reserve this name for

$$J_1=D=\lim_{\alpha,\beta=\infty}\frac{1}{\pi}\int_0^\alpha du\int_0^\beta f(t)\cos u(t-x)\,dt+\lim_{\alpha,\beta=\infty}\frac{1}{\pi}\int_0^\alpha du\int_{-\beta}^0 f(t)\cos u(t-x)dt.$$

Here $J_1$ and $J_4$ have the same relative significance as in the preceding sections. It is evident that $J_5$ does not exist. A good deal has been written * recently on the problem of finding the most general conditions that one can impose on $f(t)$ and make $J_4$ equal to $f(x)$. In the following section, it is shown that $J_1$ equals $f(x)$ under circumstances more general than those derived for $J_4$. Indeed, if $f(x)=1$, $J_4$ does not exist, and if a value be given it by the usual summation method, that value is zero, but $J_1=1$. In Section 11 the corresponding integral which arises from three-dimensional problems is considered, and some of the questions connected with it. For simplicity $x$, $y$, and $z$ are restricted to non-negative values. This enables one to dispense with the second term of $D$. No generality is lost thereby.

10. **The Fourier Double Integral.** LEMMA. *(a) For all $\alpha, \beta \geqq 0$ uniformly,*

$$\left|\int_0^\alpha \frac{\sin \beta t}{t}\,dt\right| \leqq \frac{\pi}{2}.$$

*(b) Let $0 \leqq f(t+x) \leqq M$, and let $f$ be a monotonic decreasing function of $t\,(t \geqq 0)$ for each $x$. Then, uniformly with respect to $\alpha \geqq \alpha_0 > 0$, and boundedly † with respect to $x$ in the first case, and uniformly in the second:*

$$\lim_{\beta=\infty}\int_0^\alpha f(t+x)\frac{\sin \beta t}{t}\,dt=\frac{\pi}{2}f(+0+x),\quad \lim_{\beta=\infty}\int_{\alpha_0}^\alpha f(t+x)\frac{\sin \beta t}{t}\,dt=0.$$

*(c) If also uniformly with respect to $x$ $\lim_{t=0} f(t+x)=f(+0+x)$, the convergence in the first case under (b) is uniform.*

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* A. Pringsheim, *Mathematische Annalen*, Vol. LXVIII (1910), pp. 367–408. W. H. Young, *Transactions of the Cambridge Philosophical Society*, Vol. XXI (1910), pp. 428–451; *Proceedings of the Royal Society of Edinburgh*, Vol. XXXI (1911), pp. 559–586.

† $F_n(x, y, \ldots)$ approaches its limit "boundedly" with respect to $x, y, \ldots$ if there exists a fixed number greater than $|F_n|$ uniformly for all values of $x, y, \ldots$.

The lemma is known except perhaps for the boundedness and uniformity of the convergence. By the more familiar part, however, we have

$$\frac{\pi}{2}f(+0+x)-\lim_{\beta=\infty}\int_0^a f(t+x)\frac{\sin\beta t}{t}\,dt$$
$$=\lim_{\beta=\infty}\int_0^a [f(+0+x)-f(t+x)]\frac{\sin\beta t}{t}\,dt=\lim_{\beta=\infty} S.$$

By $(a)$ and the second theorem of the mean,

$$|S|\leqq M\max_a\left|\int_0^a\frac{\sin\beta t}{t}\right|\leqq\frac{\pi M}{2}.$$

Thus the convergence is bounded. In the cases where it is uniform the demonstration is the same except that $M$ is to be chosen arbitrarily.

THEOREM 15. *Let $0\leqq f(x)\leqq M$, and let $f(x)$ be a monotonic decreasing function of $x(x\geqq 0)$. Then $D=1/2[f(x-0)+f(x+0)]$, and the proper integral of which $D$ is the limit approaches $D$ boundedly with respect to $x$ if $x\geqq a>0$.*

CASE 1: $(x=0)$. By the lemma

$$\pi D=\lim_{a,\beta=\infty}\int_0^a du\int_0^\beta f(t)\cos tu\,dt=\lim_{a,\beta=\infty}\int_0^\beta f(t)\frac{\sin\alpha t}{t}\,dt=\frac{\pi}{2}f(+0).$$

CASE 2: $(0<a\leqq x)$. In the formal expression for $D$ substitute $t-x=s$. Then

$$\pi D=\lim_{a,\beta=\infty}\int_0^a du\int_{-x}^{\beta-x} f(s+x)\cos su\,ds=\lim_{a,\beta=\infty}\int_{-x}^{\beta-x} f(s+x)\frac{\sin\alpha s}{s}\,ds$$
$$=\lim_{a,\beta=\infty}\int_{-x}^{0} f(s+x)\frac{\sin\alpha s}{s}\,ds+\lim_{a,\beta=\infty}\int_0^{\beta-x} f(s+x)\frac{\sin\alpha s}{s}\,ds.$$

Substituting $t=-s$ in the first integral, and separating the last integral into two parts by the point $x$,

$$\pi D=\lim_{a,\beta=\infty}\int_0^x [f(-t+x)+f(t+x)]\frac{\sin\alpha t}{t}\,dt+\lim_{a,\beta=\infty}\int_x^{\beta-x} f(s+x)\frac{\sin\alpha s}{s}\,ds,$$

which equals $\frac{1}{2}[f(x-0)+f(x+0)]$ by $(b)$ of the lemma, and the convergence is bounded.

COROLLARY 1. *If also $f(x)$ is continuous in $(0<a\leqq x\leqq b)$, the convergence is uniform there.*

Apply $(c)$ of the lemma.

COROLLARY 2. *The theorem and corollary are true if $f(x)=f_1(x)-f_2(x)$, where $f_1(x)$ and $f_2(x)$ have the properties there ascribed to $f(x)$.*

*Examples:* $e^{-x}, 1-e^{-x}, f(x)$ equal to a polynomial in $(0,1)$ and to $1/x$ elsewhere.

THEOREM 16. *Let* (1°) $f(x)=g(x)p(x)$, *where* $0\leqq f, g, p\leqq M$, $f(x)$ *has limited variation in every finite* interval, and* $g(x)$ *is monotonic decreasing* $(x\geqq x_0)$, *and* $p(x)$ *is periodic* $(x\geqq x_0)$. *Let* (2°), *except perhaps a null set, for* $x\geqq x_0$,

$$p(x)=\sum_{i=0}^{\infty}(a_i \sin kix+b_i \cos kix),$$

*where the* $a_i$*'s and* $b_i$*'s are the usual Fourier coefficients, and* (3°) $\Sigma(|a_i|+|b_i|)=H$. Then for each $x\geqq 0$, $D=1/2[f(x-0)+f(x+0)]$.

Let $f(t)=\phi(t)+\psi(t)$, where $\phi=f$ and $\psi=0$ at points where $t\leqq 2x$, and $\phi=0$ and $\psi=f$ where $t>2x$, $x$ being the point at which the development is to be examined. Referring to the preceding corollary we see that it holds when $\phi$ replaces $f$. Referring to the proof of the previous theorem we see that, by formal processes,

$$\pi D-\pi/2[f(x-0)+f(x+0)]=\lim_{\alpha,\beta=\infty}\int_0^x[\psi(-t+x)+\psi(t+x)]\frac{\sin \alpha t}{t}dt$$
$$+\lim_{\alpha,\beta=\infty}\int_x^{\beta-x}\psi(s+x)\frac{\sin \alpha s}{s}ds=\mathrm{I}+\mathrm{II}.$$

By definition of $\psi$, I$=0$. To prove that II$=0$ we have essentially to establish for the $f$ of our present theorem only the second part of $(b)$ of the lemma to Theorem 15, with the omission of the references to boundedness and uniformity with respect to $x$. An apparent exception occurs when $x=0$, but it may be avoided by separating the interval $(\alpha, \beta)$ into two parts by means of a fixed point. We proceed, therefore, to show that, $\alpha_0>0$, $\epsilon>0$, $x$ being prescribed, there exists a $\beta_\epsilon$ so that the following is true uniformly in $\alpha$ and $\beta$:

$$\left|\int_{\alpha_0}^{\alpha}g(t+x)p(t+x)\frac{\sin \beta t}{t}dt\right|<\epsilon, \qquad (\beta>\beta_\epsilon).$$

The function $g(t+x)/t$ is monotonic decreasing if $t>x_0$. Let $\alpha_1$ be so large that $\alpha_1>x_0$ and also that $g(t+x)/\alpha_1<\epsilon$. Since† $gp/t$ is absolutely $L$-integrable in $(\alpha_0, \alpha_1)$,

$$\lim_{\beta=\infty}\int_{\alpha_0}^{\alpha_1}\frac{gp}{t}\sin \beta t dt=0.$$

Now

$$\left|\int_{\alpha_1}^{\alpha}\frac{g(t+x)}{t}p(t+x)\sin \beta t dt\right|<\epsilon\max_{\lambda}\left|\int_{\alpha_1}^{\lambda}p(t+x)\sin \beta t dt\right|$$
$$=\epsilon\max_{\lambda}\left|\int_{\alpha_1}^{\lambda}\sin \beta t\left[\sum_0^{\infty}a_i \sin ki(t+x)+b_i \cos ki(t+x)\right]dt\right|.$$

*These conditions could be generalized a little by requiring that certain of them should hold necessarily only in the neighborhood of $x$.

† Lebesgue, *Annales de la Faculté de Toulouse*, Ser. 3, Vol. I (1909), p. 52.

It may be seen in several ways that this series is integrable termwise; for example, by 3° its remainder is uniformly limited. Thus the above expression equals

$$\epsilon \max_{\lambda} \left| \sum_{0}^{\infty} \left[ a_i \int_{a_1}^{\lambda} \sin ki(t+x) \sin \beta t dt + b_i \int_{a_1}^{\lambda} \cos ki(t+x) \sin \beta t dt \right] \right| \leqq 8\epsilon\Sigma(|a_i|+|b_i|) = 8\epsilon H.$$

COROLLARY. *The theorem is true if* $f(x)=f_1(x)-f_2(x)$, *where* $f_1$ *and* $f_2$ *separately enjoy the properties there ascribed to* $f$.

*Example:* $f=x$ in $(0,\pi)$, $=2\pi-x$ in $(\pi, 2\pi)$, etc.

THEOREM 17. *If* $f(x)$ *has limited variation in a neighborhood* $(x-c, x+c)$, $c>0$, *of* $x$, *and if* $f(x)/x$ *is absolutely L-integrable in the infinite interval* $(x \geqq c)$, *then, for each* $x \geqq 0$, $D=1/2[f(x-0)+f(x+0)]$.

As before

$$D=\lim_{\alpha,\beta=\infty} \frac{1}{\pi} \int_{-x}^{\beta-x} f(t+x) \frac{\sin \alpha t}{t} dt = \lim_{\alpha,\beta=\infty} \frac{1}{\pi} \left[ \int_{-x}^{-\frac{c}{2}} \frac{f \sin \alpha t}{t} dt + \int_{-\frac{c}{2}}^{\frac{c}{2}} \frac{f \sin \alpha t}{t} dt + \int_{\frac{c}{2}}^{\beta-x} \frac{f \sin \alpha t}{t} dt \right] = \text{I}+\text{II}+\text{III}.$$

$$\text{II}=\lim_{\alpha,\beta=\infty} \frac{1}{\pi} \int_{0}^{\frac{c}{2}} \tfrac{1}{2}[f(-t+x)+f(t+x)] \frac{\sin \alpha t}{t} dt = \tfrac{1}{2}[f(x-0)+f(x+0)],$$

since the part of the integrand in brackets has limited variation in $(0, c/2)$. Since $f(t+x)/t$ is absolutely $L$-integrable in $(-x, -c/2)$, by Lebesgue's* theorem I$=0$; and, similarly, a part of III,

$$\lim_{\alpha,\beta=\infty} \frac{1}{\pi} \int_{\frac{c}{2}}^{\beta_\epsilon} f(t+x) \frac{\sin \alpha t}{t} dt = 0,$$

where $\beta_\epsilon$ is so chosen that

$$\int_{\beta_\epsilon}^{\infty} \left| \frac{f(t+x)}{t} \right| dt < \epsilon.$$

Thus $|\text{III}| < \epsilon$.

*Example:* $f(x)=p(x)e^{-x}$, where $p$ is any limited function which has limited variation in the neighborhood of $x$ and is $L$-integrable in every finite interval.

It could now be shown that, if in this form of the Fourier integral the integrand were multiplied by $e^{-ay}$ (or by $e^{-c^2a^2t}$), the usual thermostatic (or thermodynamic) equations would be satisfied, in general, and that the functions

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* *Loc. cit.*

defined thereby would enjoy the other properties desired for the solutions of the corresponding problems in heat and electricity. But this would not be useful, for, although some texts imply that the Fourier "double" (in my sense, iterated) integral is needed in these physical problems, it never is, and the appropriate propositions can be proved for the simple integrals that are needed in a much more general fashion than I can prove them even for the double integrals considered here.* The theorems of this section serve as lemmas to the next section, where their practical value becomes apparent. However, as notably in the case of Fourier's series,† so to a less degree in the case of Fourier's integral, the theory of the representation of an arbitrary function has enjoyed a development and interest that have nothing to do with its value as a physical tool.

11. **The Fourier Quadruple Integral.** If one seeks the value of the potential function $V$ in an infinite rectangular parallelopiped $(x, y, z \geqq 0)$, given that the surface $z=0$ is kept at a potential $V_0=f(x, y)$, and that the surface $x=0$ is insulated, one is led to LaPlace's equation, $\nabla^2(V)=0$, and to the formula: ‡

$$V_z = \lim_{\alpha, \beta, \gamma, \delta=\infty} \int_0^\alpha dt \int_0^\beta du \int_0^\gamma ds \int_0^\delta e^{-z\sqrt{s^2+w^2}} f(t, u) \cos s(t-x) \cos w(u-y)\, dw.$$

THEOREM 18. *Let $f(x, y)=h(x)k(y)$, where $h$ and $k$ satisfy the conditions imposed on $f$ in one of the theorems* 15, 16, *or* 17, *or are the differences of two such functions. Then* $V_0=1/4[f(x-0, y-0)+f(x-0, y+0)+f(x+0, y-0)+f(x+0, y+0)]$.

For $V_0 = \lim_{\alpha, \beta, \gamma, \delta} \left[\int_0^\alpha dt \int_0^\gamma h(t) \cos s(t-x)\, ds\right]\left[\int_0^\beta du \int_0^\delta k(y) \cos w(u-y)\, dw\right]$, which equals $1/4[h(x-0)+h(x+0)][k(y-0)+k(y+0)]$.

COROLLARY. *The convergence is uniform with respect to $x$ (or $y$) if $h$ (or $k$) also satisfies the condition of continuity imposed in the corollary to Theorem* 15.

THEOREM 19. *If $f(x, y)$ is absolutely L-integrable in the infinite rectangle $(x, y \geqq 0)$, $V_z(z>0)$ satisfies LaPlace's equation.*

It is sufficient to show that the signs of partial differentiation may be passed over the limit sign and over the signs of integration in the expression for $V_z$. For this purpose we refer to corollary $(b)$ of Theorem 14. Using

*Cf. W. H. Young, *Proceedings of the Royal Society of Edinburgh*, Vol. XXXI (1911), p. 587; *Encyklopädie Mathematischen Wissenschaften*, II, A 12, p. 1089.

† Cf. VanVleck, "The Influence of Fourier's Series upon the Development of Mathematics," *Science*, N. S., Vol. XXXIX (1914), Number 995, pp. 113–124.

‡ In the usual formula the iterated integral is used.

the notation there, but replacing that $f$ by the present integrand, which we call $\phi$, and thinking of $n$ as representing in turn $x$, $y$, and $z$, we see that we need to prove: (1) that $J_1(d\phi/dn)$ is continuous near $n$, $n$ being the point at which the derivative of $J_1$ is being examined; (2) that $J_1(\phi)$ exists at $n$; (3) that $d\phi/dn$ exists, and is absolutely $L$-integrable over finite three-dimensional fields; and (4) that there exists a function, $\Phi(n)$, whose integral with respect to $n$ exists over a finite interval enclosing the particular point $n$ in question, such that, for all values of $\alpha, \beta, \gamma, \delta \geqq 0$ uniformly,

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{d\phi}{dn}\, dt\, du\, ds\, dw \right| < \Phi(n).$$

These results are needed in order to show that the first derivative of $J_1$ may be replaced by the first derivative of its integrand. The process must be repeated for the second derivative. We shall then have conditions (5), (6), (7), and (8) exactly like (1), (2), (3), and (4), respectively, except that $d\phi/dn$ replaces $\phi$, and in (4) $\Phi(n)$ is replaced by some other function having the same properties. It is obvious that (3) and (7) are true, and that (6) is included in (1). We first prove (4). For all values of $x$, $y$, $z$ uniformly,

$$|\phi| \leqq |f(t,u)|\, e^{-s\sqrt{s^2+w^2}} \leqq |f|\, e^{-s\left(\frac{s+w}{2}\right)} = |f|\, e^{-\frac{zs}{2}}\, e^{-\frac{zw}{2}},$$

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \phi \right| \leqq M \left( \int_0^\infty e^{-\frac{zs}{2}}\, ds \right)^2 = \frac{4M}{z^2},$$

where

$$M = \lim_{\alpha,\beta} \int_0^\alpha dt \int_0^\beta |f(t,u)|\, du.$$

Thus (2) $J_1(\phi)$ exists absolutely. Now, first let $n=z$.

$$\left.\begin{aligned} &\frac{\partial\phi}{\partial z} = -\sqrt{s^2+w^2}\,\phi, \quad \frac{\partial^2\phi}{\partial z^2} = (s^2+w^2)\phi, \quad \left|\frac{\partial\phi}{\partial z}\right| \leqq |f|\,(s+w)\, e^{-\frac{zs}{2}}\, e^{-\frac{zw}{2}}, \\ &\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{\partial\phi}{\partial z} \right| \leqq 2M \left( \int_0^\infty s e^{-\frac{zs}{2}}\, ds \int_0^\infty e^{-\frac{zw}{2}}\, dw \right) = \frac{16M}{z^3}. \end{aligned}\right\} \quad (9)$$

Thus (4) is satisfied ($z>0$). We can similarly prove (8), for

$$\left| \int_{R_{\alpha,\beta,\gamma,\delta}} \frac{\partial^2\phi}{\partial z^2} \right| \leqq 2M \int_0^\infty s^2 e^{-\frac{zs}{2}}\, ds \int_0^\infty e^{-\frac{zw}{2}}\, dw = \frac{64M}{z^4}.$$

Now, by referring to Corollary 1 ($d^\circ$) of Theorem 12, it will be seen that in (9) we have established (1) and (5); *e. g.*, if $z \geqq c > 0$,

$$|f|\,(s+w)\, e^{-\frac{z}{2}(s+w)} \leqq |f|\,(s+w)\, e^{-\frac{c}{2}(s+w)},$$

which is an integrable function in $R$. Secondly, let $n=x\geqq 0$.

$$\frac{\partial\phi}{\partial x} = se^{-z\sqrt{s^2+w^2}}f(t,u)\sin s(t-x)\cos w(u-y),$$

$$\frac{\partial^2\phi}{\partial x^2} = -s^2e^{-z\sqrt{s^2+w^2}}\cos s(t-x)\cos w(u-y).$$

$$\left|\int_{R_{\alpha,\beta,\gamma,\delta}}\frac{\partial\phi}{\partial x}\right| < M\int_0^\infty se^{-\frac{zs}{2}}\,ds\int_0^\infty e^{-\frac{zw}{2}}\,dw = \frac{8M}{z^3}.$$

$$\left|\int_{R_{\alpha,\beta,\gamma,\delta}}\frac{\partial^2\phi}{\partial x^2}\right| < M\int_0^\infty s^2e^{-\frac{zs}{2}}\,ds\int_0^\infty e^{-\frac{zw}{2}}\,dw = \frac{32M}{z^4}.$$

Hence the conditions in question are satisfied. The case, $n=y\geqq 0$, is identical with this one.

WESLEYAN UNIVERSITY, MIDDLETOWN, CONN., *October*, 1916.